Canadian Association of Amateur Astronomers (CAAA)

Canadian Astronomy and Astrophysics Olympiads

CAAO tutorial

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Preface

This guide is intended for students who wish to participate in the Canadian Astronomy and Astrophysics Olympiad (CAAO). It serves as an introductory-level introduction to the Olympiad for interested students across Canada. Each year, the highest-achieving CAAO students are selected to represent Team Canada in the International Astronomy and Astrophysics Olympiad (IOAA), and are provided with a training program to prepare them for the international competition.

While this guide contains a great deal of information, we recommend that students supplement their learning with other resources listed in the reference section. The guide includes numerous practice problems designed to complement students' learning path. Additionally, we highly encourage students to solve past CAAO problems available in a separate file.

To participate in the Canadian astronomy Olympiad, students should have a solid foundation in high school-level physics and mathematics. However, for the international Olympiads, students will need to develop an advanced understanding of physics and mathematics beyond what is typically taught in high school.

The International Astronomy and Astrophysics Olympiad (IOAA) is a prestigious international competition for high school students. Each year, the brightest students from around the world compete in this event, and Canada has been participating in the IOAA since 2013.

Resources

The Canadian Astronomy and Astrophysics Olympiad (CAAO) requires diligent preparation by interested students, and the use of appropriate resources is critical to success. Several textbooks have been identified as valuable resources in this endeavor, including:

1. Foundations of Astrophysics, authored by Barbara Ryden
2. Fundamental Astronomy, written by Karttunen et al.
3. An Introduction to Modern Astrophysics, co-authored by Bradley Carroll and Dale Ostlie
4. Astronomy Principles and Practice, by Archie E. Roy and David Clarke
5. Introduction to Cosmology, authored by Barbara Ryden

The first two textbooks are introductory level, while An Introduction to Modern Astrophysics is suitable for students with a strong background in physics and calculus. Students seeking comprehensive knowledge of spherical astronomy are advised to reference Astronomy Principles and Practice.

Additionally, students are expected to possess a solid foundation in high school-level physics and mathematics. Senior-level students are strongly encouraged to deepen their understanding of these subjects by studying calculus.

Aspiring students may benefit from exploring more advanced IOAA-level resources. The references used to compile this document have been included in the reference section for this purpose.

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PART I

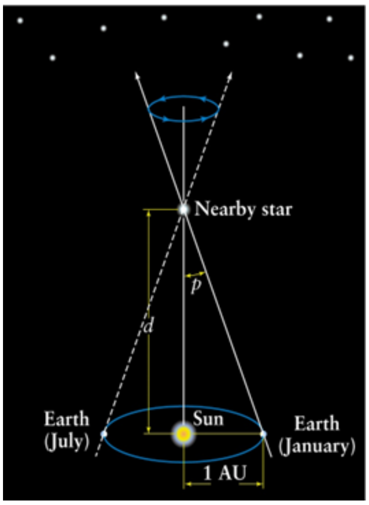
Introduction to CAAO

CHAPTER 1

Basic Concepts

1.1 Parallax

Measuring the intrinsic brightness of stars is linked with determining their distances. On Earth, the distance to the peak of a remote mountain can be determined by measuring that peak's angular position from two observation points separated by a known baseline distance. Simple trigonometry then supplies the distance to the peak. Finding the distance even to the nearest stars requires a longer baseline. As Earth orbits the Sun, two observations of the same star made 6 months apart employ a baseline equal to the diameter of Earth's orbit. These measurements reveal that a nearby star exhibits an annual back-and-forth change in its position against the stationary background of much more distant stars. a measurement of the parallax angle p (one-half of the maximum change in angular position) allows the calculation of the distance d to the star.



We can write the equation as:

$$d = \frac{1AU}{\tan p} \approx \frac{1}{p} AU \tag{1.1}$$

The angle p is smaller as the distance becomes larger. Using parallax, we are going to introduce a new distance measure, parsec.
A parsec is the distance at which 1 Astronomical Unit subtends an angle of 1 second of arc (arcsecond):

$$1 \; parsec = 3.26 \; light \; years$$

$$1pc = \frac{1AU}{\left(\frac{1}{3600} \times \frac{\pi}{180}\right)} = 206265AU$$

Figure 1.1: Parallax triangle

1 arc-second is 1/60 of an arc-minute, and an arc-minute is 1/60 of one degree. Therefore, an arc-second is 1/3600 th of one degree.

1.2 Stellar Luminosity

Stars are considered as a spherical source of radiating energy due to their temperature; their total energy output can be determined by the equation below, according to their surface temperature and surface area. This total output is referred to as the stellar luminosity, L, and may be expressed as:

$$L = 4\pi R^2 \sigma T^4, \tag{1.2}$$

where R is the radius of the star, σ is known as **Stefan-Boltzmann's constant** and T is the star's surface temperature. The unit of luminosity is Watts (joules per second). For instance, the luminosity of our sun is $3.85 \times 10^{26}\ W$.

1.3 Brightness (radiant flux)

The brightness of a star is measured in terms of the flux received from the star. The power received per unit area at the Earth depends on the stellar luminosity and on the inverse square of the stellar distance. If the latter is known, the flux provided by the source may be readily calculated and expressed in terms of watts per square metre (W/m^2). Imagine a star of luminosity L surrounded

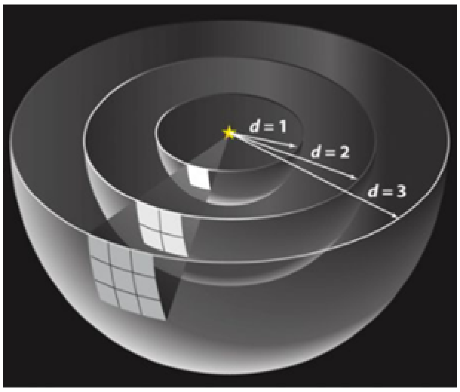


Figure 1.2: Flux vs. distance

by a huge spherical shell of radius d. Then, assuming that no light is absorbed during its journey out to the shell, the radiant flux, b, measured at distance d is related to the star's luminosity by:

$$b = \frac{L}{4\pi d^2}, \tag{1.3}$$

the denominator is simply the area of the sphere. Since L does not depend on d, the radiant flux is inversely proportional to the square of the distance from the star. This is the well-known inverse square law for light.

1.4 Magnitude system

Invented by the astronomer Hipparchus 2200 years ago, it was simply a way to "rank" the stars visible at night. The brightest were ranked as 1st magnitude, the faintest visible were ranked as 6th magnitude. In other words, the brightest stars were assigned the smallest number, the faintest the largest number. And 6 divisions were used because of the mysticism about 6, which is the first **perfect number**. The brightness ratio of rank first and sixth is 100:

$$K^5 = 100 \rightarrow K = \sqrt[5]{100} = 2.5118 \rightarrow \tfrac{b_2}{b_1} = 2.5118^{m_1 - m_2}$$

$$m_1 - m_2 = -2.5 \log \frac{b_1}{b_2}, \tag{1.4}$$

Where m is the magnitude of the stars. The magnitude system is based on the comparison; This means that you need to know the magnitude of a certain star and by comparing its brightness with other stars you can determine the magnitude of the star.

The absolute magnitude, M, is defined to be the apparent magnitude a star would have if it were located at 10 pc. Recall that a difference of 5 magnitudes between the apparent magnitudes of two stars corresponds to the smaller magnitude star being 100 times brighter than the larger-magnitude star. This allows us to find an equation for the absolute magnitude just like the apparent magnitude:

$$M_1 - M_2 = -2.5 \log \frac{L_1}{L_2}, \tag{1.5}$$

Where M is the absolute magnitude and L is the luminosity of the star. To use this equation, we need to know a specific star's absolute magnitude and luminosity to be able to compare it with other stars. We have defined two different magnitudes, absolute and apparent. We defined absolute magnitude as the apparent magnitude at a certain distance (10 parsecs), therefore, there should be a connection between the star distances and their magnitudes. This relation is called **Distance Modulus**:

$$m - M = 5 \log d - 5, \tag{1.6}$$

Where d is in parsecs, m and M are apparent and absolute magnitudes respectively. Unlike the previous magnitude relations (Equations 1.4-1.5), distance modulus is written for a single star. It is the relationship between the absolute and apparent magnitude of any star in the distance d. Using this relation, it is clear that if you have any star in the distance of 10 $parsecs$, then the two magnitudes should be equal to each other.

1.5 Limiting magnitude

Looking up at the night sky, we are not able to see all the stars; our eyes have limited light-gathering aperture (around 6mm on a dark night!). The faintest stars that your naked eye can see in the night are about $6 - 6.5$ magnitudes. However, this limiting magnitude might also change due to light pollution or atmospheric effects. For instance, in a metropolitan area, your limiting magnitude might go up to $2 - 3$ magnitudes. This would seriously limit your ability to see constellations.

We know that using optical devices would enable us to see fainter objects in the sky. For instance, if you are using a telescope since it has a bigger diameter, it is able to gather more light than your eyes can. Therefore, we have the equation below to determine the limiting magnitude of a telescope:

$$m_e - m_t = -5 \log \frac{D_t}{D_e}, \tag{1.7}$$

Where m_e is the limiting magnitude of the naked eye ($m_e \approx 6.5$), m_t is the telescope's limiting magnitude, D_e is the pupil's diameter ($D_e \approx 6\ mm$) and D_t is the telescope's diameter.

CHAPTER 2

Telescopes

2.1 Optical Telescopes

An optical telescope forms images of faint and distant stars. It can collect much more light from space than the human eye can. Optical telescopes are built in two basic designs—**refractors** and **reflectors**. The heart of a telescope is its objective, a main lens (in refractors) or a mirror (in reflectors). Its function is to gather light from a sky object and focus this light to form an image. The ability of a telescope to collect light is called its **light-gathering power**.

Light-gathering power is proportional to the area of the collecting surface, or to the square of the **aperture** (clear diameter of the main lens or mirror). The size of a telescope, such as 150 *mm* or 8 *m* (6-*inch* or 26-foot), refers to the size of its aperture. You can look at the image directly through an eyepiece, which is essentially a magnifying glass. Or you can photograph the image or record and process it electronically. Your eye lens size is about 6 *mm*. A 150 *mm* (6-*inch*) telescope has an aperture over 30 times bigger than your eye lens. Its light-gathering power is 900 times greater than that of your eye. So, a star appears over 900 times brighter with a 150 *mm* (6-*inch*) telescope than it does to your unaided eye.

Astronomers build giant telescopes to detect fainter and more distant objects. All stars appear brighter with telescopes than they do to the eye alone. The extra starlight gathered by the telescope is concentrated into a single point. Using time exposure, a giant 10 *m* (400-*inch*) telescope can image very faint stars down to about magnitude 28, which is the same apparent brightness as a candle viewed from the Moon!

2.2 Refracting Telescopes

A refracting telescope has a main, objective lens permanently mounted at the front end of a tube. Starlight enters this lens and is refracted, or bent so that it forms an image near the back of the tube. The distance from this lens to the image is its focal length. You may look at the image through a removable magnifying lens called the eyepiece. The tube keeps out scattered light, dust, and moisture. Italian astronomer Galileo Galilei (1564–1642) first pointed a refracting telescope skyward in 1609. The largest instrument he made was smaller than 50 *mm* (2-*inches*).

Today refracting telescopes range in size from a beginner's 60-*mm* (2.4-*inch*) to the largest ever built, the 1 *m* (40-*inch*) telescope at the Yerkes Observatory in Williams Bay, Wisconsin, U.S., which was completed in 1897.

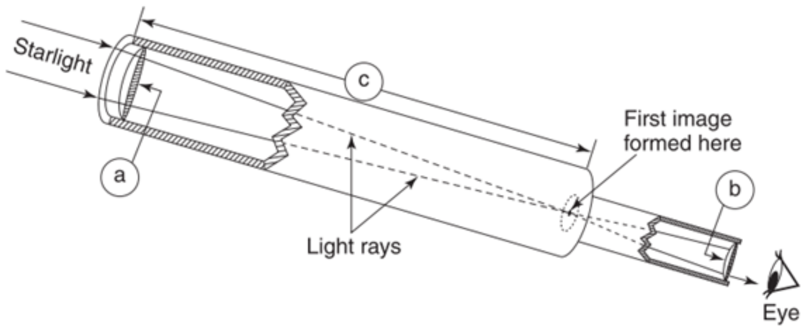


Figure 2.1: (a) Objective lens gathers the light and forms an image. (b) Eyepiece magnifies the image formed by the objective. (c) The focal length of the objective lens.

2.3 Reflecting Telescopes

A **reflecting telescope** has a highly polished curved-glass mirror, the **primary mirror**, mounted at the bottom of an open tube. When starlight shines on this mirror, it is reflected back up the tube to form an image at the **prime focus**.

You can record the image at the prime focus, or you can use additional mirrors to reflect the light to another spot. The **Newtonian telescope**, originated by British scientist Sir Isaac Newton in 1668, uses a small, flat mirror to reflect the light through the side of the tube to an eyepiece (Figure below). The **Cassegrain telescope** uses a small convex mirror, a *secondary mirror*, to reflect the light back through a hole cut in the primary mirror at the bottom end of the tube. It is more compact than a refractor or Newtonian reflector of the same aperture. The **Schmidt-Cassegrain** telescope combines an extremely short-focus spherical primary mirror at the back end of a sealed tube with a thin lens at the front.

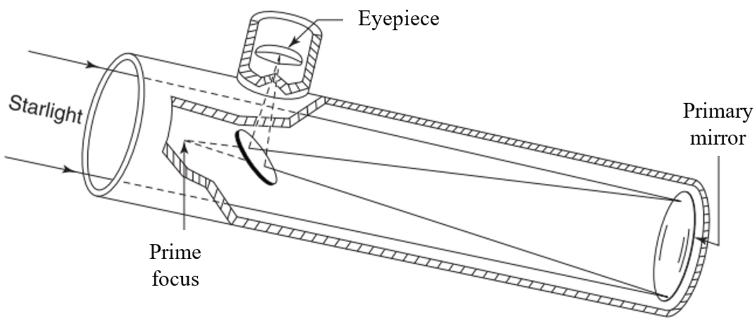


Figure 2.2: Newtonian reflecting telescope

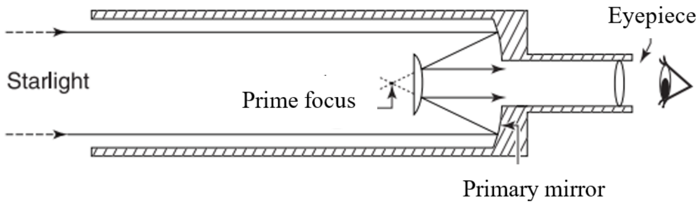


Figure 2.3: Cassegrain reflecting telescope.

2.4 F-number

Telescopes are often described by both their aperture size and **f-number**. The f-number is the ratio of the focal length of the main lens or mirror to the aperture. These specifications are important because the brightness, size, and clarity of the image produced by a telescope depend on the aperture and focal length of its main lens or mirror. For example, a "150-*mm* (6-*inch*), $f/8$ reflector" means the primary mirror is 150 *mm* (6-*inches*) in diameter and has a focal length of 1200 *mm* (8×150), or 48 inches (8×6).

2.5 Images

All stars except our Sun are so far away that they appear as dots of light in a telescope. The Moon and planets appear as small disks. **Image size** is proportional to the focal length of the telescope's main lens or mirror.

For example, a mirror with a focal length of 2.5 *m* (100 *inches*) produces an image of the Moon that measures about 2.5 *cm* (1 *inch*) across. You know that the 5 *m* (200-*inch*), $f/3.3$ mirror has a focal length of 16.5 *m* (660-*inches*), which is over six times as long. Hence, it produces an image of the Moon that is about six times as big or 15 *cm* (6-*inches*) across.

Lenses and mirrors form real images that are upside down. (A real image is formed by the actual convergence of light rays.) Since inverted images do not matter in astronomical work and righting them would require additional light-absorbing optics, nothing is done to turn images upright in telescopes.

2.6 Lens and the focal length

In a camera with a lens, the image will be in focus only at a fixed distance F from the lens. The distance F to the **focal plane** depends on the shape of the lens, as well as on its refractive index. For lenses, a useful parameter is the focal ratio $f = F/D$, where D is the diameter of the lens. The size of the image produced is not affected by the diameter D of the lens but only by the focal length F.

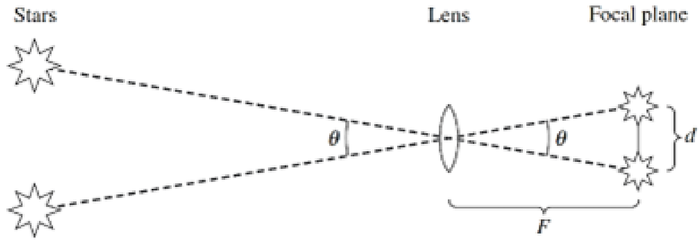


Figure 2.4: Focal plane.

In the $figure$ 2.4, two stars separated by a small angle θ on the sky have images that are separated by a physical distance d on the focal plane. Another useful parameter, in addition to the focal length, is the scale of the image on the focal plane, known for historical reasons as the plate scale. Specifically, an angular distance θ on the celestial sphere is related to a physical distance d on the image plane by the plate scale s:

$$\theta[arcsec] = s[arcsec/mm].d[mm], \tag{2.1}$$

we can also write this as:

$$\theta[radians] = \frac{d}{F}, \tag{2.2}$$

and therefore:

$$\theta[arcsec] = \theta[radians].\frac{180°}{\pi[radians]}.\frac{3600arcsec}{1°} = 206265(\frac{d}{f}), \tag{2.3}$$

we have a relationship between the plate scale s and the focal length F:

$$s[arcsec/mm] = \frac{206265}{F[mm]}. \tag{2.4}$$

The human eye, for instance, has a focal length $F \approx 17\ mm$, and hence a "plate scale" $s \approx 12,100\ arcsec/mm$, or $s \approx 3.4°/mm$; when you look at the full Moon, its image covers an area of your retina less than 0.15 mm across. Large astronomical telescopes have focal lengths that are more conveniently expressed in meters than in millimeters.

For these big telescopes, we may write:

$$s[arcsec/mm] = \frac{206.265}{F[m]} = \frac{206.265}{fD[m]}, \tag{2.5}$$

where f is the focal ratio, and D is the diameter of the telescope's aperture. As an example, the famous "forty-inch" Yerkes Telescope (at Williams Bay, Wisconsin) has an aperture $D = 1.02\ m$ and a focal ratio $f = 19$. The plate scale of the Yerkes Telescope is thus:

$$s = \frac{206.265}{19 \times (1.02)} arcsec/mm = 10.6\ arcsec/mm, \tag{2.6}$$

therefore an image of the full Moon produced by the Yerkes Telescope is 170 mm across, about the size of a salad plate.

The major optical component of a *refracting telescope* is the primary or objective lens of focal length f_{obj}. The purpose of the objective lens is to collect as much light as possible and with the greatest possible resolution, bringing the light to a focus at the focal plane. A photographic plate or other detector may be placed at the focal plane to record the image, or the image may be viewed with an eyepiece, which serves as a magnifying glass. The eyepiece would be placed at a distance from the focal plane equal to its focal length, f_{eye}, causing the light rays to be refocused at infinity. The figure below shows the path of rays coming from a point source lying off the optical axis at an angle θ. The rays ultimately emerge from the eyepiece at an angle ϕ from the optical axis. The angular magnification produced by this arrangement of lenses can be shown to be:

$$m = \frac{f_{obj}}{f_{eye}}. \tag{2.7}$$

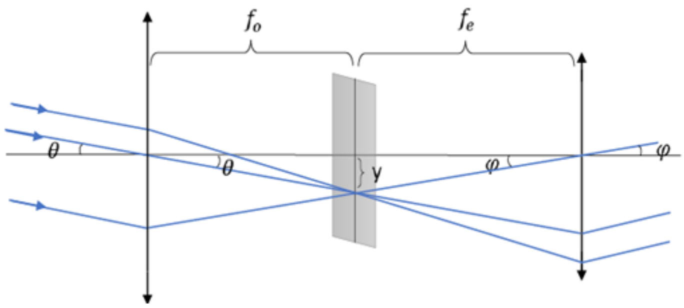


Figure 2.5: Telescope Magnification

In astronomy, the field of view is the amount of sky you can see, whether with your unaided vision, binoculars, or a telescope. If you had eyes on all sides of your head, you would have a 360° field of view. (Some insects actually do!) If you include peripheral vision, your naked eye field of view is nearly 180°, but with varying quality across this field. A telescope will have a much smaller field of view, but it has significant advantages, such as greater magnification and light-gathering power.

Field of view (FOV) is the diameter of a region of the sky that you can see using a specific instrument. The FOV would change with the magnification of the telescope you are using.

Figure 2.6: Field of view

2.7 Telescope resolving power

The **resolving power** of a telescope is defined as the ability of a telescope to distinguish objects with a small angle between them. This is known as the **theoretical resolving power** of the instrument. If the telescope is of good design and in correct adjustment, it should be possible to achieve this theoretical value. It should be possible to resolve two stars if they are separated by an angle (in radians) greater than:

$$\alpha = 1.22\frac{\lambda}{D}. \tag{2.8}$$

This value is known as the theoretical angular resolving power of the telescope. It can be seen that the resolving power is inversely proportional to the diameter of the objective. We take a value of 5500 *Angstroms* for λ being the effective wavelength for visual observations.

CHAPTER 3

Observing the Universe

3.1 Stellar Evolution

Stellar evolution refers to the life cycle of a star, from its formation through to its eventual demise. The study of stellar evolution is important not only for understanding the structure and behavior of individual stars, but also for understanding the evolution of galaxies and the universe as a whole. In this article, we will discuss the various stages of stellar evolution, including the formation of stars, the main sequence phase, the evolution of different types of stars, and their eventual fates.

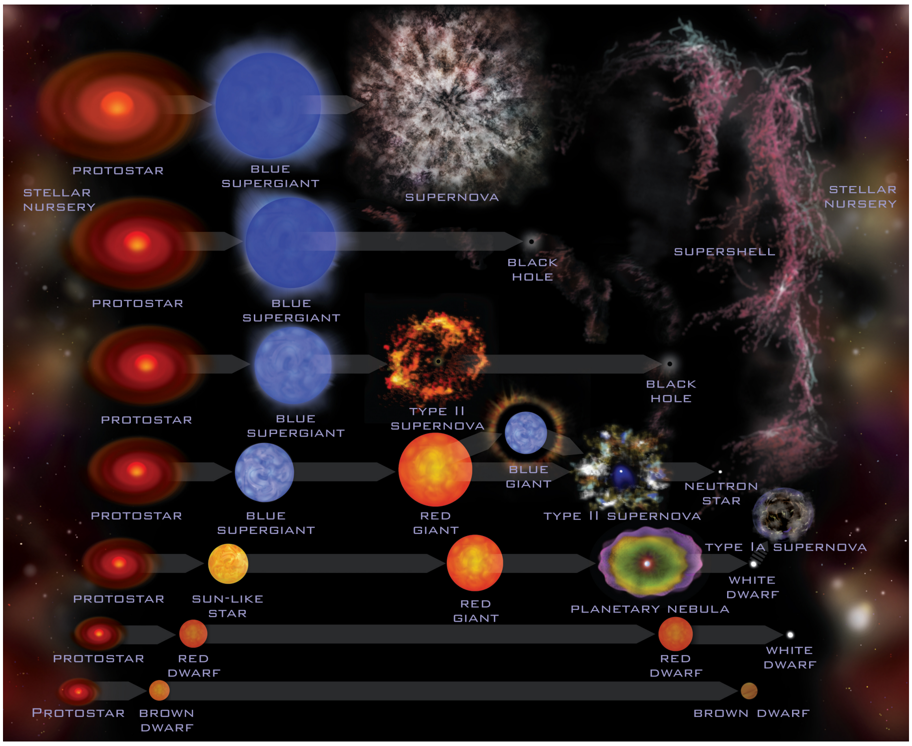


Figure 3.1: Stellar evolution for stars with different initial masses

Formation of Stars

Stars are formed from large clouds of gas and dust, known as **nebulae**. These clouds are primarily composed of hydrogen and helium, with smaller amounts of heavier elements. The process of star formation begins when a region of a nebula becomes dense enough for gravity to take over. As the gas and dust collapse under their own weight, the temperature and pressure at the center of the cloud begin to rise. Eventually, the temperature and pressure become high enough to trigger nuclear fusion, and a new star is born.

Main Sequence Phase

The main sequence is the longest phase in a star's life cycle. During this phase, the star is in a state of hydrostatic equilibrium, which means that the inward pull of gravity is balanced by the outward pressure created by nuclear fusion in the core. The temperature and pressure at the core of the star are high enough to fuse hydrogen atoms into helium, releasing energy in the process.

The size, luminosity, and color of a star during its main sequence phase depend on its mass. More massive stars are hotter, brighter, and bluer than less massive stars. The Sun, for example, is a main sequence star with a mass of about 1.99 x 10^{30} kg.

Red Dwarfs

Red dwarfs are the most common type of star in the universe. They are small, cool stars with masses less than about 0.5 times that of the Sun. Because they are so small and cool, red dwarfs can burn hydrogen for a very long time, with some estimated to live up to trillions of years.

White Dwarfs

White dwarfs are the remnants of low-mass stars, such as red dwarfs or main sequence stars with masses less than about 8 times that of the Sun. When these stars run out of fuel, they no longer have the outward pressure from nuclear fusion to balance the inward pull of gravity, and they collapse inward. This collapse causes the outer layers of the star to be expelled in a planetary nebula, leaving behind a hot, dense core known as a white dwarf.

Giants and Supergiants

Giants and supergiants are stars that have exhausted the hydrogen fuel in their cores and have begun fusing heavier elements. As the core contracts and heats up, the outer layers of the star expand and cool, causing the star to increase in size and luminosity. Giants and supergiants are classified based on their size and luminosity, with supergiants being the largest and most luminous.

Supernovae

Supernovae are some of the most violent events in the universe. They occur when a massive star runs out of fuel and can no longer generate enough pressure to resist the inward pull of gravity. The core of the star collapses, creating a shock wave that causes the outer layers of the star to explode outward in a brilliant display of light and energy. This explosion can briefly outshine an entire galaxy and release more energy than the Sun will produce in its entire lifetime.

There are two types of supernovae: Type I and Type II. Type I supernovae occur when a white dwarf in a binary star system accretes enough matter from its companion to reach a critical mass and undergo a runaway fusion reaction. Type II supernovae occur when a massive star runs out of fuel and collapses, triggering a shock wave that causes the outer layers of the star to explode.

Black Holes and Neutron Stars

When a massive star collapses in on itself, the core can become so dense that it forms a black hole or a neutron star. Black holes are regions of space where the gravity is so strong that nothing, not even light, can escape. Neutron stars, on the other hand, are incredibly dense, with the mass of the Sun compressed into a sphere only about 20 km in diameter.

Figure 3.2 is called a black hole shadow. The black hole's extreme gravity alters the paths of light coming from different parts of the disk, producing the warped image. The black hole's extreme gravitational field redirects and distorts light coming from different parts of the disk, but exactly what we see depends on our viewing angle. The greatest distortion occurs when viewing the system nearly edgewise.

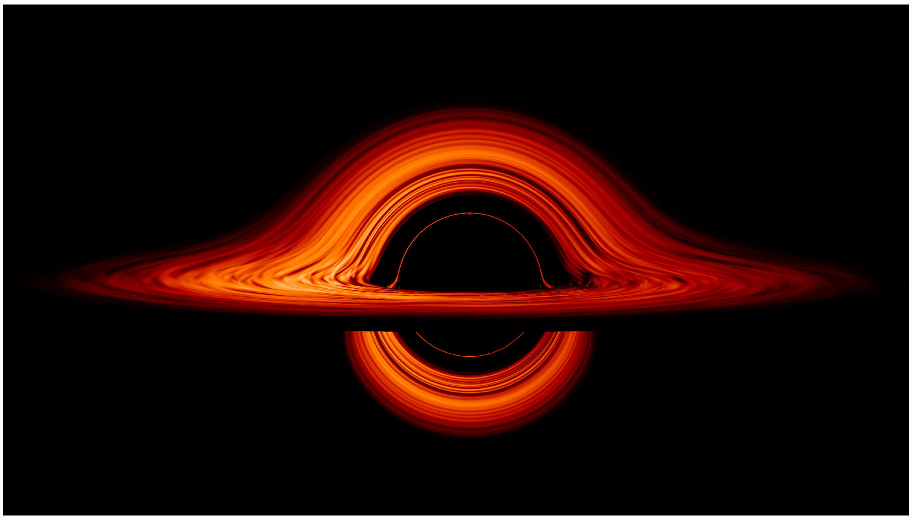


Figure 3.2: The turbulent disk of gas churning around a black hole takes on a crazy double-humped appearance

Stellar evolution is a complex process that is still not fully understood. However, by studying the different stages of stellar evolution, astronomers can gain a better understanding of the physical processes that govern the universe.

3.2 Galaxies

Galaxies are vast systems of stars, gas, and dust held together by gravity. They come in a variety of shapes and sizes, and play a crucial role in our understanding of the universe. We will explore the different types of galaxies and their properties, as well as some fascinating facts about these cosmic structures.

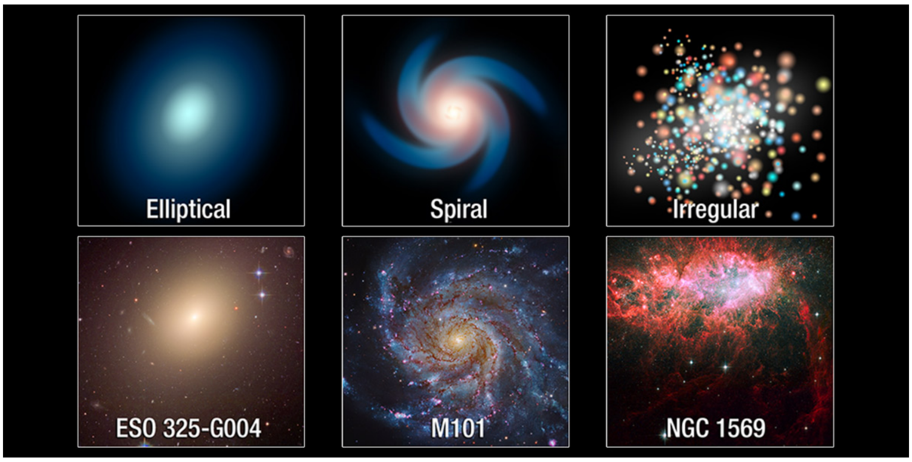


Figure 3.3: Different types of galaxies

Elliptical Galaxies

Elliptical galaxies are the most common type of galaxy in the universe. They are shaped like ellipsoids and contain mostly old, red stars. Elliptical galaxies can range in size from small dwarf galaxies to giant ellipticals that are up to 20 times larger than the Milky Way.

The largest known elliptical galaxy, IC 1101, has a diameter of over 6 million light years and contains trillions of stars.

Spiral Galaxies

Spiral galaxies are characterized by their prominent spiral arms, which are made up of young, blue stars, gas, and dust. The Milky Way is a spiral galaxy, and our Sun is located in one of its spiral arms. Spiral galaxies come in a variety of shapes, from tight and compact to loose and open.

The Andromeda Galaxy, our closest neighboring galaxy, is a spiral galaxy that is expected to collide with the Milky Way in about 4 billion years.

Irregular Galaxies

Irregular galaxies have no well-defined shape and contain a mix of old and young stars, gas, and dust. They are often small and found in the vicinity of larger galaxies, which can disrupt their structure through tidal forces.

The Small Magellanic Cloud, a satellite galaxy of the Milky Way, is an irregular galaxy that is visible to the naked eye from the southern hemisphere.

Size and Mass

Galaxies come in a range of sizes and masses, with elliptical galaxies typically being the largest and most massive, and irregular galaxies being the smallest and least massive. The mass of a galaxy is typically measured by its rotation curve, which describes the speed of stars and gas as a function of their distance from the center.

The smallest known galaxy, $Segue2$, has a mass of only about 1,000 times that of the Sun and contains just a few hundred stars.

Composition

Galaxies are composed of a variety of elements, including hydrogen, helium, and heavier elements such as carbon and oxygen. The relative abundances of these elements can provide clues about the formation and evolution of a galaxy.

The stars in the Andromeda Galaxy contain more heavy elements than those in the Milky Way, suggesting that it has had a more active history of star formation.

Black Holes

Many galaxies, including the Milky Way, contain supermassive black holes at their centers. These black holes can have masses ranging from millions to billions of times that of the Sun and can profoundly influence the evolution of their host galaxies.

The largest known black hole, TON 618, has a mass of about 66 billion times that of the Sun and is located in a quasar, an extremely luminous object powered by material falling onto the black hole.

Galaxies are incredibly diverse and complex systems that continue to fascinate astronomers and the public alike. By studying their properties and evolution, we can gain insights into the history and structure of the universe.

Galaxy Evolution

Galaxies are not static objects, but rather are constantly changing over time. The study of galaxy evolution seeks to understand how galaxies form, grow, and change over time.

Galaxies are thought to have formed from the gravitational collapse of primordial gas clouds in the early universe. These clouds were primarily composed of hydrogen and helium, with small amounts of heavier elements. As the gas clouds collapsed, the temperature and pressure at the center of the cloud increased, triggering the formation of the first stars and galaxies.

Over time, galaxies grow by merging with other galaxies. When galaxies merge, their stars and gas clouds interact gravitationally, causing them to lose energy and fall towards the center of the newly-formed galaxy. As more and more galaxies merge together, the resulting galaxy becomes larger and more massive.

Galaxies also evolve through the process of star formation. As stars form and die within a galaxy, they release heavy elements into the surrounding gas. Over time, the concentration of heavy elements within a galaxy increases, leading to changes in the way the gas clouds behave. This can lead to changes in the rate of star formation within the galaxy.

Observations of distant galaxies suggest that they were more active in the past than they are today. Galaxies in the early universe were more likely to be forming stars at a rapid rate and were often much more irregular in shape than galaxies we see today. This suggests that galaxies have undergone significant evolution over time.

Galaxy evolution is a complex and ongoing area of research in astronomy. By studying the properties of galaxies at different points in cosmic history, astronomers hope to gain a better understanding of how galaxies form and evolve over time, and how they contribute to the overall structure and evolution of the universe.

3.3 Doppler effect and Red shift

In 1842 the Austrian physicist Christian Doppler showed that as a source of sound moves through a medium (such as air), the wavelength is compressed in the forward direction and expanded in the backward direction. This change in wavelength of any type of wave caused by the motion of the source or the observer is called a Doppler shift. Doppler deduced that the difference between the wavelength λ_{obs} observed for a moving source of sound and the wavelength λ_{rest} measured in the laboratory for a reference source at rest is related to the radial velocity v_r (the component of the velocity directly toward or away from the observer, of the source through the medium by:

$$\frac{\lambda_{obs} - \lambda_{rest}}{\lambda_{rest}} = \frac{\Delta\lambda}{\lambda_{rest}} = \frac{v_r}{v_s}, \tag{3.1}$$

where v_s is the speed of sound in the medium. When astronomers observe a star or galaxy moving away from or toward Earth, the wavelength of the light they receive is shifted toward longer or shorter wavelengths, respectively. If the source of light is moving away from the observer ($v_r > 0$), then $\lambda_{obs} > \lambda_{rest}$. This shift to a longer wavelength is called a redshift. Similarly, if the source is moving toward the observer ($v_r < 0$), then there is a shift to a shorter wavelength, a blueshift. most of the objects in the universe outside of our own Milky Way Galaxy are moving away from us, redshifts are commonly measured by astronomers. A redshift parameter z is used to describe the change in wavelength; it is defined as:

$$z = \frac{\lambda_{obs} - \lambda_{rest}}{\lambda_{rest}} = \frac{\Delta\lambda}{\lambda_{rest}}. \tag{3.2}$$

There is a relation between the velocity of the object moving away or toward us and the redshift z. If the velocity of the object is comparable to the speed of light, we need to use the relativistic formula:

$$z = \sqrt{\frac{1 + \frac{v_r}{v_s}}{1 - \frac{v_r}{v_s}}} - 1, \tag{3.3}$$

Where c is the speed of light. Its non-relativistic formula can be written as:

$$z = \frac{v_r}{c} \tag{3.4}$$

3.4 Hubble's Law

In the late 1920s, Hubble discovered that the spectral lines of galaxies were shifted towards the red by an amount proportional to their distances. If the redshift is due to the Doppler effect, this means that the galaxies move away from each other with velocities proportional to their separations (this means that the Universe is expanding as a whole). Hubble's law states that the redshift in the light coming from distant galaxies is proportional to their distance. The discovery of the linear relationship between recessional velocity and distance yields a straightforward mathematical expression for Hubble's Law as follows:

$$v = H_0 D, \tag{3.5}$$

where v is the recessional velocity due to redshift, typically expressed in km/s, D is the distance of the object in Mpc, and H_0 is Hubble's constant. Usually, the Hubble's constant is around $65 - 75\ \frac{km}{s.Mpc}$ in the problems.

3.5 Planetary motion

The apparent motions of the planets are quite complicated, partly because they reflect the motion of the Earth around the Sun. Normally the planets move eastward (direct motion, counterclockwise as seen from the Northern hemisphere) when compared with the stars.

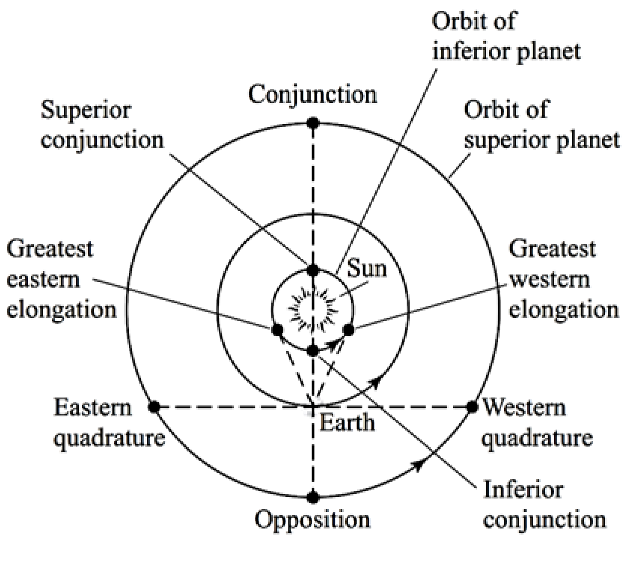


Figure 3.4: Planetary configurations

The fact that Mercury and Venus are never seen more than 28° and 47°, respectively, east or west of the Sun clearly shows that their orbits are located inside the orbit of Earth. These planets are referred to as **inferior planets**, and their maximum angular separations east or west of the Sun are known as the greatest eastern elongation and greatest western. elongation, respectively (see *Figure* 3.4).

Opposition occurs when the Earth lies between the Sun and the superior planet. That is, the Sun and planet are 180° apart on the celestial sphere as seen from the Earth. A **conjunction** occurs when the Sun lies between the Earth and the superior planet. That is, the Sun and planet are 0° apart as seen from the Earth. **Quadrature** occurs when the Sun and the superior planet are 90° apart as seen from the Earth. The quadrature can be either eastern, when the planet appears 90° east of the Sun in the sky, or western when the planet appears 90° west of the Sun. **Inferior conjunction** occurs when the inferior planet lies between the Earth and the Sun. **Superior conjunction** occurs when the Sun lies between the Earth and the inferior planet.

The relative orbital motions of Earth and the other planets mean that the time interval between successive oppositions or conjunctions can differ significantly from the amount of time necessary to make one complete orbit relative to the background stars. The former time interval (between oppositions) is known as the *synodic period* (S), and the latter time interval (measured relative to the background stars) is referred to as the *sidereal period* (P). The relationship between the two periods is given by:

$$\frac{1}{S} = \frac{1}{T_P} - \frac{1}{T_E}, \tag{3.6}$$

where T_P is the period of rotation of a random planet, T_E is the period of Earth (1 *year*), and S is the *synodic period* (or the period between two successive same situations of the two planets).

CHAPTER 4

Spherical Astronomy

We have seen that the observer who views the heavens at night gets the impression that they are at the centre of a great hemisphere onto which the heavenly bodies are projected. The moon, planets, and stars seem to lie on this celestial hemisphere, their directions defined by the positions they have on its surface. For many astronomical purposes, the distances are irrelevant so the radius of the sphere can be chosen at will. The description of the positions of bodies on it, considering positional changes with time, necessarily involves the use of special coordinate and timekeeping systems. The relationship between the positions of bodies requires a knowledge of the geometry of the sphere. This branch of astronomy, called **spherical astronomy**, is in one sense the oldest branch of the subject, its foundations dating back at least 4000 years. Its subject matter is still essential and never more so than today when the problem arises of observing or calculating the position of an artificial satellite or interplanetary probe. We, therefore, begin by considering the geometry of the sphere.

4.1 Spherical geometry

The geometry of the sphere is made up of great circles, small circles, and arcs of these figures. Distances along great circles are often measured as angles since, for convenience, the radius of the sphere is made unity. A great circle is defined to be the intersection with the sphere of a plane containing the centre of the sphere. Since the centre is equidistant from all points on the sphere, the figure of intersection must be a circle. If the plane does not contain the centre of the sphere, its intersection with the sphere is a small circle.

We can draw infinite circles on a sphere, some may have a radius of the sphere (great circles) and others will have a smaller radius (small circles). In the figure on the right ANBM, CNDM, and APBQ are all great circles, while EFG is a small circle.

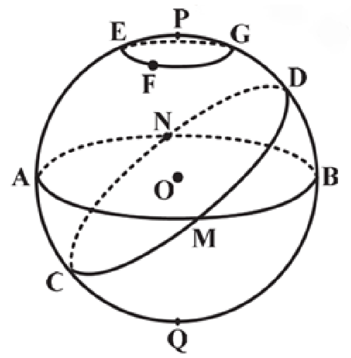


Figure 4.1: Great and small circles on a sphere

The area of the spherical triangle can be found by the equation:

$$S_{ABC} = (A + B + C - \pi)R^2, \tag{4.1}$$

where all angles must be written in terms of *Radians* and R is the radius of the sphere. Just as the formulas of plane trigonometry can be used to perform calculations in plane geometry, special trigonometrical formulas for use in spherical geometry can be established. There are many such formulas but four are more often used than any of the others. They are the relations between the sides and angles of a spherical triangle and are invaluable in solving the problems that arise in spherical astronomy.

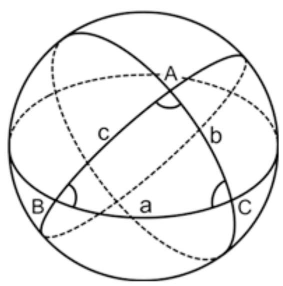


Figure 4.2: Area of a spherical triangle with angles of A, B, and C

ABC is a spherical triangle with sides AB, BC, and CA of lengths c, a, and b, respectively, and with angles $\angle CAB$,$\angle ABC$, and $\angle BCA$ hereafter referred to as angles A, B and C respectively. The four formulas are:

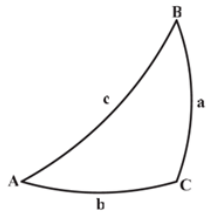


Figure 4.3: Sample spherical triangle

Sine formula:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \tag{4.2}$$

Cosine formula:

$$\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A \\ \cos b &= \cos a \cos c + \sin a \sin c \cos B \\ \cos c &= \cos a \cos b + \sin a \sin b \cos C \end{aligned} \tag{4.3}$$

Polar formula formula:

$$\begin{aligned} -\cos A &= \cos B \cos C + \sin B \sin C \cos a \\ -\cos B &= \cos A \cos C + \sin A \sin C \cos b \\ -\cos C &= \cos A \cos B + \sin A \sin B \cos c \end{aligned} \tag{4.4}$$

Four-parts formula:

$$\cos a \cos C = \sin a \cot b - \sin C \cot B \tag{4.5}$$

4.2 Position on the Earth's surface

To illustrate these concepts, we consider the Earth. Geographers have already shown us how to set up a coordinate system on a sphere; the system of **latitude** and **longitude** provides a coordinate system on the surface of the (approximately) spherical Earth. On the Earth, the north and south poles represent the points where the Earth's rotation axis passes through the Earth's surface. The **equator** is the great circle midway between the north and south pole, dividing the Earth's surface into a northern hemisphere and a southern hemisphere.

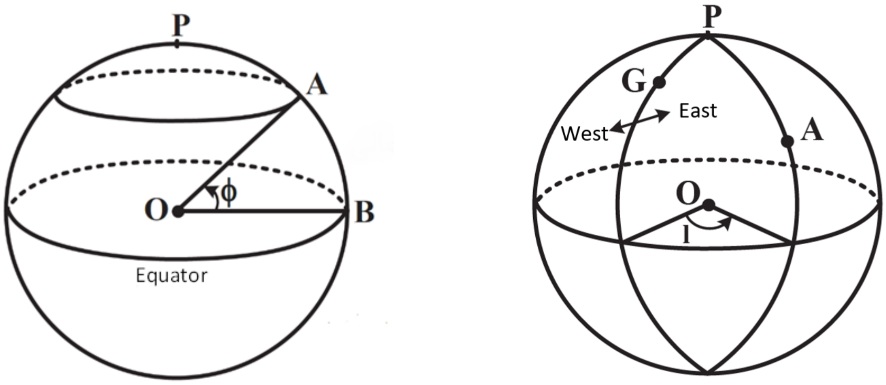


Figure 4.5: Longitude l

The latitude of a point on the Earth's surface is its angular distance from the equator, measured along a great circle perpendicular to the Earth's equator. Latitude is measured in degrees, arc-minutes, and arc-seconds, as is longitude. Thus, the use of latitude and longitude does not require knowing the size of the Earth in kilometers or any other unit of length. The longitude may be expressed in angular measure or in time units related to each other by the table on the right.

$360° = 24^{h}$	
$1° = 4^{m}$	$1^{h} = 15°$
$1' = 4^{s}$	$1^{m} = 15'$
$1'' = (1/15)^{s}$	$1^{s} = 15''$

Figure 4.6: Unit conversions used in spherical astronomy

4.3 The horizontal (alt-azimuth) system

It is convenient to imagine a sphere at a great distance ("infinity") upon which all stars lie. This is called the **celestial sphere**. The positions of stars on this sphere may be specified with two angles, analogous to the way latitude and longitude specify a position on the earth's surface. This celestial sphere is an artificial construction; stars are not all at the same distance. Stars in our own Galaxy range in the distance from 4 light-years to more than 50000 light-years from the Earth. Nevertheless, the concept of the celestial sphere is useful for charting the sky as one sees it.

One such coordinate system on the celestial sphere is based on an observer's horizon and hence is called the horizon coordinate system. In this system, the latitude-like coordinate is the altitude, defined as the angle of a celestial object above the horizon circle. The zenith (the point directly overhead) is at an altitude of 90°. Points on the horizon circle are at an altitude of 0°. The nadir is at an altitude of −90°, but in practice, negative altitudes are seldom used, since they represent objects that are hidden by the Earth. The longitude-like coordinate in the horizon coordinate system is called the azimuth

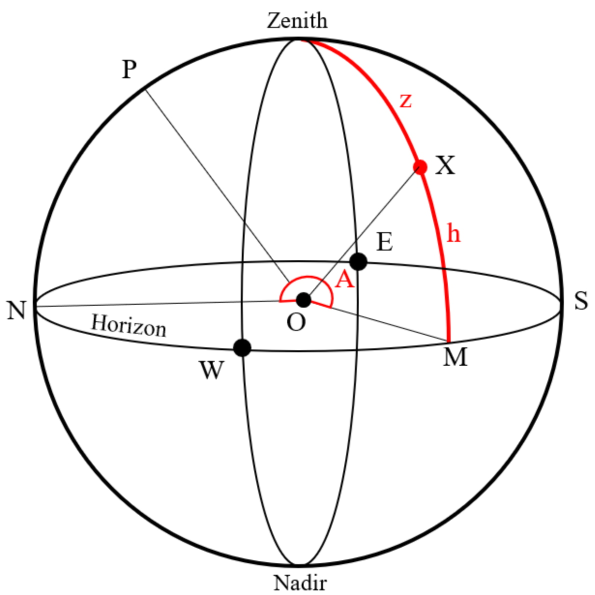


Figure 4.7: Horizontal(alt-azimuth) coordinate system

In the above figure, X is the position of the star. The arc of $\widehat{XM} = h$, where h is the altitude. Therefore, the zenith distance is $\widehat{XZ} = 90° - h = z$. The azimuth of this star is the red angle shown in the figure: $A = 360° - \widehat{NOM}$. Azimuth is usually expressed from North to East. But if the star is located in the western hemisphere (like the star in the figure), we can express the azimuth from North to West: $A = \widehat{NOM}\, W$.

For any point on the celestial sphere, half a great circle can be drawn from the zenith, through the point in question, to the nadir. The half-circle that runs through the north point on the horizon circle acts as the ı*primemeridian*ȷ in the horizon coordinate system. The azimuth is measured in degrees running from north to east. An object due north of an observer has an azimuth of 0°, an object due east has an azimuth of 90°, and so forth. If you know the altitude and azimuth of any object in your horizon coordinate system, you know where to point your telescope to see it.

If we consider the figure below for an observer in a particular latitude of ϕ, the direction of rotation of the Earth is P_1, and since the north celestial pole (NCP) is in distant, P_2 will be the direction for that the person will see the north celestial pole. It is depicted that the altitude of the pole is equal to the latitude of the observer.

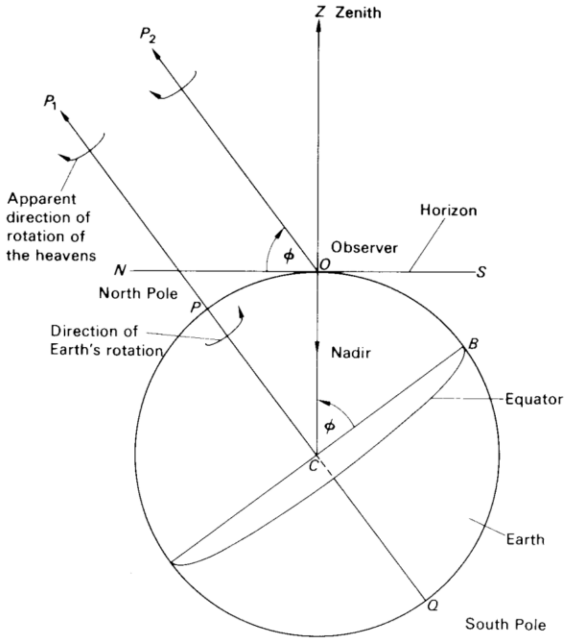


Figure 4.8: altitude of NCP for the observer on Earth

One shortcoming of the horizon coordinate system is that every observer on Earth has a different, unique horizon and hence has a different, unique horizon coordinate system. A star that is near the zenith ($altitude \approx 90°$) for an observer in Buenos Aires will be near the nadir (altitude -90°) for an observer in the antipodal city of Shanghai. To describe the positions of objects on the celestial sphere, it is useful to have a coordinate system that all astronomers, regardless of location, can agree on, just as geographers all agree to use latitude and longitude to describe positions on the Earth.

4.4 The equatorial system

To build a coordinate system that works for everyone on Earth, we start by projecting the Earth's poles and equator outward onto the celestial sphere. The Earth's rotation axis, which passes through the north and south poles of the Earth, intersects the celestial sphere at the **north celestial pole** (labeled as NCP) and the **south celestial pole** (labeled as SCP). The north celestial pole is at the zenith for an observer at the Earth's north pole; more generally, for an observer at a latitude north of the equator, it will be at an altitude of ϕ and azimuth of 0°. The projection of the Earth's equator onto the celestial sphere is called the **celestial equator**. The celestial equator passes through the zenith for an observer on the Earth's equator.

On the Earth's surface, a point's latitude is its angular distance north or south of the equator. Similarly, on the celestial sphere, a point's **declination** (δ) is its angular distance north or south of the celestial equator. For points north of the celestial equator, the declination is positive ($0° < \delta \leq 90°$), and for points south of the celestial equator, the declination is negative ($-90° \leq \delta < 0°$).

Right ascension α is analogous to longitude and is measured eastward along the celestial equator from the vernal equinox (γ) to its intersection with the object's hour circle (the great circle passing through the object being considered and through the north celestial pole). Right ascension is traditionally measured in hours, minutes, and seconds. The coordinates of the right ascension and declination are also indicated in the figure below. Since the equatorial coordinate system is based on the celestial equator and the vernal equinox, changes in the latitude and longitude of the observer do not affect the values of right ascension and declination. Values of and are similarly unaffected by the annual motion of Earth around the Sun.

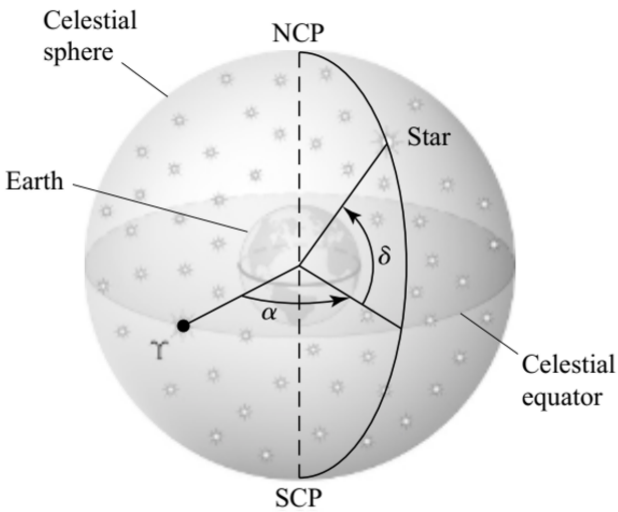


Figure 4.9: Equatorial coordinate system

We used a unique point to define the Azimuth angle in horizontal coordinates, that specific point is North. Having that in mind, we also know that the NCP is pointing towards the North. We can draw both coordinates on a sphere for an observer.

If we sketch both coordinates on a single sphere, then the celestial equator intersects the horizon circle in two points West and East. Points P and Z are the poles of the celestial equator and the horizon respectively. But W lies on both these great circles so that W is 90° from the points P and Z. Hence, W is a pole on the great circle ZPN and must, therefore, be 90° from all points on it—in particular from N and S. Hence, it is the west point. By a similar argument, E is the east point. Any great semicircle through P and Q is called a *meridian*. The meridian through the celestial object X is the great semicircle $PXBQ$ cutting the celestial equator in B.

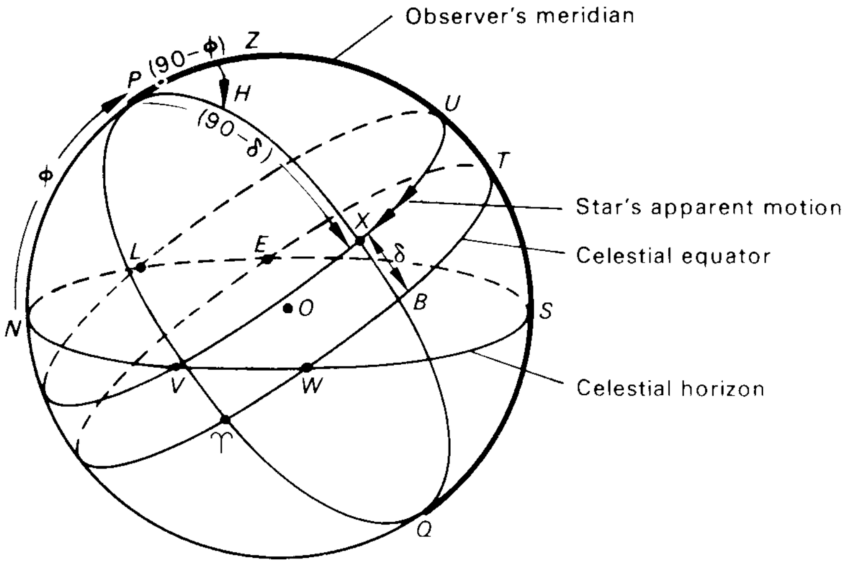


Figure 4.10: Equatorial coordinate system for an observer on Earth

In particular, the meridian $PZTSQ$ indicated because of its importance by a heavier line is the **observer's meridian**. An observer viewing the sky will note that all natural objects rise in the east, climbing in altitude until they **transit** across the observer's meridian then decrease in altitude until they set in the west. A star, in fact, will follow a small circle parallel to the celestial equator in the arrow's direction. Such a circle (UXV in the diagram) is called a **parallel of declination** and provides us with one of the two coordinates in the equatorial system. The **declination**, δ, of the star is the angular distance in degrees of the star from the equator along the meridian through the star. It is measured north and south of the equator from 0° to 90°, being taken to be positive when north. The declination of the celestial object is thus analogous to the latitude of a place on the Earth's surface, and indeed the latitude of any point on the surface of the Earth when a star is in its zenith is equal to the star's declination.

A quantity called the **north polar distance** of the object (X in the figure) is often used. It is the arc PX.

Obviously,

$$\textit{north polar distance} = 90^\circ - \textit{declination}.$$

It is to be noted that the north polar distance can exceed 90°. The star, then, transits at U, sets at V, rises at L and transits again after one rotation of the Earth. The second coordinate recognizes this. The angle ZPX is called the **hour angle**, t, of the star and is measured from the observer's meridian westwards (for both north and south hemisphere observers) to the meridian through the star from 0^h to 24^h or from 0° to 360°. Consequently, the hour angle increases by 24^h each sidereal day for a star. Having both coordinates on the sphere, using Zenith, the North celestial pole, and the star (three points) we are able to create a spherical triangle (figure below). We need to use spherical trigonometry to solve any spherical triangle.

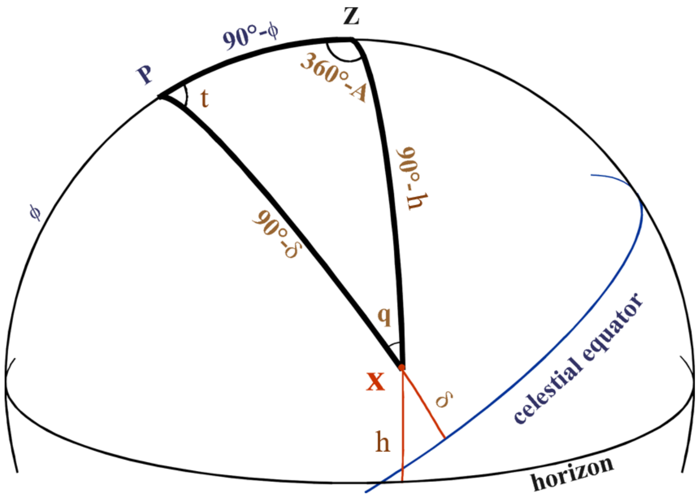


Figure 4.11: Coordinate system conversion

A common problem in spherical astronomy is obtaining a star's coordinates in one system, given the coordinates in another system. The observer's latitude is usually known. For example, we may want to calculate the hour angle of t and declination δ of a body when its azimuth (east of north) and altitude are A and h. Assume the observer has a latitude ϕ. We start by writing the cosine formula:

$$\begin{aligned} &\cos PX = \cos PZ \cos ZX + \sin PZ \sin ZX \cos PZX \\ \longrightarrow\quad &\sin\delta = \sin\phi \sin h + \cos\phi \cos h \cos A \end{aligned} \tag{4.6}$$

By using the cosine formula again:

$$\begin{aligned} &\cos ZX = \cos PZ \cos PX + \sin PZ \sin PX \cos ZPX \\ \longrightarrow\quad &\sin a = \sin\phi \sin\delta + \cos\phi \cos\delta \cos t \end{aligned} \tag{4.7}$$

You could also use four-parts or sine law to solve the spherical triangle. Based on the known parameters in the triangle, you should decide which formulas to use in order to solve the triangle.

4.5 The ecliptic system

The orbital plane of the Earth, the *ecliptic*, is the reference plane of another important coordinate frame. The ecliptic can also be defined as the great circle on the celestial sphere described by the Sun over the course of one year. This frame is used mainly for planets and other bodies of the solar system. The orientation of the Earth's equatorial plane remains invariant, unaffected by its annual motion. In spring, the Sun appears to move from the southern hemisphere to the northern one. The time of this remarkable event as well as the direction of the Sun at that moment is called the vernal equinox. At the *vernal equinox*, the Sun's right ascension and declination are zero.

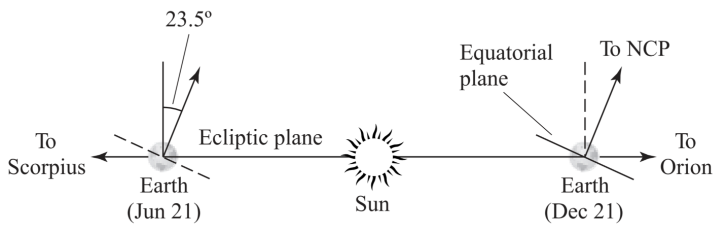


Figure 4.12: The plane of Earth's orbit seen edge-on

The two quantities specifying the position of an object on the celestial sphere in this system are ecliptic longitude and ecliptic latitude. In *figure 4.13* below a great circle arc through the pole of the ecliptic K and the celestial object X meets the ecliptic in point D. Then the **ecliptic longitude**, λ, is the angle between γ and D, measured from 0° to 360° along the ecliptic in the eastwards direction, that is in the direction in which right ascension increases. The **ecliptic latitude**, β, is measured from D to X along the great circle arc DX, being measured from 0° to 90° north or south of the ecliptic. It should be noted that the north pole of the ecliptic, K, lies in the hemisphere containing the north celestial pole. It should also be noted that ecliptic latitude and longitude are often referred to as **celestial latitude** and **longitude**.

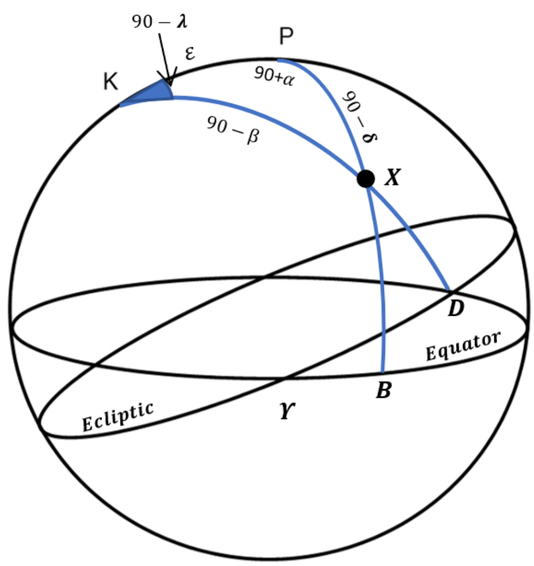


Figure 4.13: The Celestial sphere used for coordinate system conversion

Let's assume the equatorial coordinates of a star are known, and we want to determine its ecliptic coordinates. This means α and δ are given. Using the spherical triangle above, we can use cosine law:

$$\begin{aligned} &\cos(90-\beta) = \cos\epsilon\cos(90-\delta) + \sin\epsilon\sin(90-\delta)\cos(90+\alpha) \\ \longrightarrow\quad &\sin\beta = \cos\epsilon\sin\delta - \sin\epsilon\cos\delta\cos\alpha \end{aligned} \tag{4.8}$$

By using the cosine formula again:

$$\begin{aligned} &\cos(90-\delta) = \cos\epsilon\cos(90-\beta) + \sin\epsilon\sin(90-\beta)\cos(90-\lambda) \\ \longrightarrow\quad &\sin\delta = \cos\epsilon\sin\beta - \sin\epsilon\cos\beta\cos\lambda \\ \longrightarrow\quad &\sin\lambda = \tfrac{\sin\delta-\cos\epsilon\sin\beta}{\sin\epsilon\cos\beta} \end{aligned} \tag{4.9}$$

CHAPTER 5

Celestial Mechanics

By applying Newton's laws of motion and the law of universal gravitation, we are able to comprehend and analyze the complex movements of celestial objects within the solar system. The celestial objects that we can observe include the planets, comets, natural satellites, and man-made satellites that are orbiting around their respective planets. The analytical process is simplified by making two assumptions. The first assumption is that we only consider the gravitational force between the orbiting body, such as the Earth, and the central body, which is the Sun. We disregard the gravitational forces exerted by other celestial bodies, such as other planets, to focus solely on the interaction between the orbiting and central body. Secondly, we assume that the central body is significantly more massive than the orbiting body, enabling us to disregard the central body's motion caused by their mutual attraction. Although both objects actually orbit around their common center of mass, if one of the celestial bodies is much more massive than the other, the center of mass can be approximated to be at the center of the heavier object.

5.1 Newton's Law of Gravitation

Gravitational force is one of the four fundamental forces of nature, along with electromagnetic force, weak nuclear force, and strong nuclear force. It is the force that causes objects with mass to be attracted to one another. In this article, we will explore the concept of gravitational force and its mathematical description.

The English physicist Sir Isaac Newton was the first to describe the nature of gravitational force. He formulated his law of gravitation in 1687, which states that the force of attraction between two objects with masses m_1 and m_2, separated by a distance r, is given by:

$$F_G = G\frac{m_1 m_2}{r^2}, \tag{5.1}$$

where F_G is the gravitational force, G is the gravitational constant, and r is the distance between the two objects. The value of G is approximately 6.674×10^{-11} N m^2/kg^2.

Gravitational Field

The gravitational force can also be described in terms of a gravitational field. A gravitational field is a region of space where an object with mass experiences a force due to the presence of another object with mass. The gravitational field strength at a point in space is defined as the force per unit mass experienced by a small test mass placed at that point.
The gravitational field strength at a distance r from a point mass M is given by:

$$g = \frac{GM}{r^2}, \tag{5.2}$$

where G is the gravitational constant. The gravitational field strength is a vector quantity, pointing towards the point mass M.

Gravitational Potential Energy

Gravitational force is a conservative force, meaning that the work done by the force in moving an object from one point to another is independent of the path taken. The gravitational potential energy of an object at a point in space is the amount of work required to move the object from an infinite distance to that point, against the gravitational force.
The gravitational potential energy U of an object of mass m at a distance r from a point mass M is given by:

$$U = -\frac{GMm}{r}, \tag{5.3}$$

The negative sign indicates that the gravitational force is attractive, and the potential energy is lower at closer distances.

5.2 Linear Momentum

Linear momentum, also known as linear motion, is the product of an object's mass and its velocity. It is a vector quantity, meaning it has both magnitude and direction. The formula for linear momentum is:

$$p = mv, \tag{5.4}$$

where p is the linear momentum, m is the mass of the object, and v is its velocity.
Linear momentum is conserved in an isolated system, meaning that the total linear momentum of a system remains constant if no external forces act upon it. This principle is known as the law of conservation of linear momentum.

5.3 Angular Speed

Angular speed is the rate at which an object rotates or revolves about a fixed axis. It is a scalar quantity, meaning it has magnitude but no direction. The formula for angular speed is:

$$\omega = \frac{\theta}{t}, \tag{5.5}$$

where ω is the angular speed, θ is the angular displacement of the object, and t is the time taken for the object to complete the rotation.
Angular speed is measured in radians per second (rad/s). It is important to note that angular speed is not the same as linear speed, which is the distance traveled per unit time.

5.4 Angular Momentum

Angular momentum is the rotational equivalent of linear momentum. It is the product of an object's moment of inertia and its angular velocity. The formula for angular momentum is:

$$L = I\omega, \tag{5.6}$$

where L is the angular momentum, I is the moment of inertia of the object, and ω is its angular velocity.
Angular momentum can also be expressed as the product of the mass of the object, its tangential velocity, and the distance from the axis of rotation:

$$L = mrv, \tag{5.7}$$

where m is the mass of the object, v is its tangential velocity, and r is the distance from the axis of rotation.
Angular momentum is also a vector quantity, meaning it has both magnitude and direction. Its direction is perpendicular to the plane of rotation. The moment of inertia is a measure of an object's resistance to rotational motion and depends on both the mass and the distribution of mass relative to the axis of rotation.

Like linear momentum, angular momentum is conserved in an isolated system. This principle is known as the law of conservation of angular momentum. The law states that if no external torques act upon an isolated system, the total angular momentum of the system remains constant. Mathematically, this can be expressed as:

$$\frac{dL}{dt} = \tau_{net}, \tag{5.8}$$

where dL/dt is the rate of change of angular momentum and τ_{net} is the net external torque acting on the system. If there is no net external torque, then dL/dt is zero and the angular momentum of the system is conserved.

5.5 Conservation of Angular Momentum

The law of conservation of angular momentum has many important applications in physics. For example, it can be used to explain the behavior of spinning tops, the motion of planets around the sun, and the behavior of particles in quantum mechanics.

One important application of conservation of angular momentum is in understanding the behavior of rotating systems. For example, when an ice skater pulls their arms in, their moment of inertia decreases, causing their angular velocity to increase, and their angular momentum to remain constant. This principle is also used in designing objects such as satellites and gyroscopes, which rely on the conservation of angular momentum to maintain their stability and orientation in space.
Another important application of conservation of angular momentum is in the study of collisions. When two objects collide, their angular momentum may change due to external torques, such as friction. However, if the collision is elastic and there are no external torques, the total angular momentum of the system will remain constant.

Overall, the law of conservation of angular momentum is a fundamental principle in physics that helps to explain the behavior of rotating systems and the interactions between objects in motion.

5.6 Kepler's laws

Kepler's laws, which describe the motions of planets, were originally deduced by Johannes Kepler from observations of the planet Mars. However, these laws can also be derived from Isaac Newton's laws of motion and his law of gravitation, which provide an empirical basis for understanding planetary motion. Additionally, the application of Newton's laws of motion and law of universal gravitation extends beyond just the study of celestial bodies in our solar system. These laws can be applied to the study of the universe as a whole, including the behavior of galaxies and the evolution of the universe itself. They also have practical applications, such as in the design and operation of spacecraft and satellites. By understanding how gravity works and how objects move in space, scientists and engineers can plan and execute space missions with incredible precision, including everything from sending probes to explore distant worlds to placing satellites in orbit for communication and navigation purposes.

1. Kepler's First Law: All planets follow elliptical orbits with the Sun at one of the two foci. Newton realized that there is a direct mathematical relationship between inverse-square ($\frac{1}{r^2}$) forces and elliptical orbits. Figure 4.1 illustrates a typical elliptical orbit, where the orbiting body is located at polar coordinates (r, θ) and the origin is at the central body. An elliptical orbit is characterized by two parameters: the semimajor axis a and the eccentricity e. The distance from the center of the ellipse to either focus is ea. A circular orbit is a special case of an elliptical orbit with $e = 0$, where the two foci merge to a single point at the center of the circle. For example, Earth follows an elliptical orbit with an eccentricity of approximately 0.0167.
The maximum distance R_a of the orbiting body from the central body is indicated by the prefix $apo-$ (or sometimes $ap-$), as in aphelion (the maximum distance from the Sun) or apogee (the maximum distance from Earth). Similarly, the closest distance R_p is indicated by the prefix $peri-$, as in perihelion or perigee. As you can see from Figure 4.1:

$$R_a = a(1 + e), R_p = a(1 - e). \tag{5.9}$$

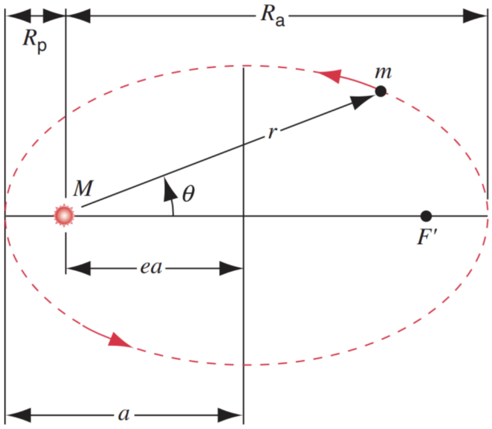


Figure 5.1: A planet of mass m moving in an elliptical orbit around the Sun with mass M

And for circular orbits $R_a = R_p$.

2. The Law of Areas: dictates that, during equal intervals of time, the imaginary line that connects a planet to its central star will cover equal areas. Figure 4.2 serves to visually demonstrate this concept, and implies that an orbiting object will move with greater velocity when it is nearer to the central body than when it is further away. It can be proven that the Law of Areas is in fact equivalent to the Law of Conservation of Angular Momentum.

If we examine the small area increment A that is traversed during a time interval t, as illustrated in Figure 4.2, we can see that the area of the triangular wedge is roughly equal to half of its base, $r\Delta\theta$ multiplied by its height r. We can then calculate the rate at which this area is swept out:

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{2} r^2 \frac{\Delta\theta}{\Delta t} = \frac{1}{2} r^2 \omega \tag{5.10}$$

If we make the assumption that the more massive body M can be considered stationary, then the angular momentum of the orbiting body m can be described relative to the origin at the central body as:

$$L_z = I\omega = mr^2\omega \tag{5.11}$$

Thus:

$$\frac{dA}{dt} = \frac{L_z}{2m} \tag{5.12}$$

If the M and m system is isolated and there is no external torque acting on it, then the angular momentum L_z remains constant. This means that the derivative of the area A with respect to time t is also constant, as stated in the equation. Consequently, during each interval of time dt, the line connecting m and M sweeps out an equal area dA, which confirms Kepler's second law. The increase in speed of a comet as it passes close to the Sun is an example of this effect and is directly related to the law of conservation of angular momentum.

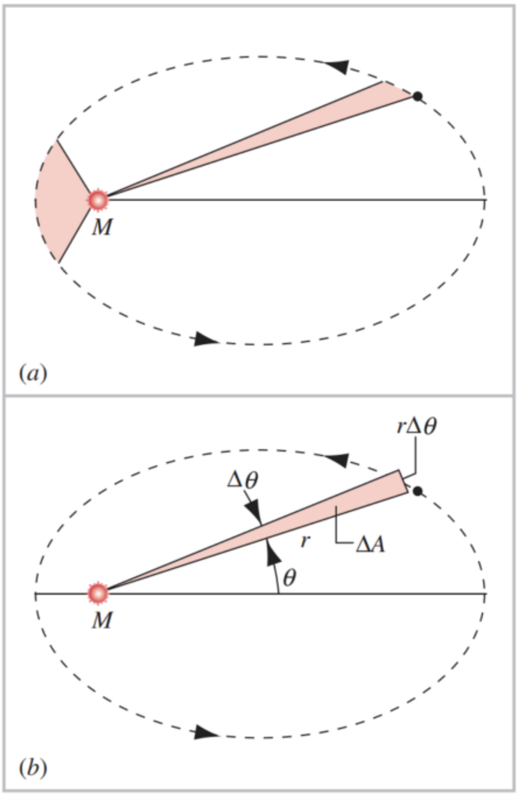


Figure 5.2: (a) The law of areas is demonstrated by the equal shaded areas, which are traversed by a line connecting a planet to the Sun in equal time intervals. (b) During a time interval t, the line connecting a planet to the Sun sweeps through an angle theta (θ) while covering an area A.

2. The Law Periods: One of the fundamental laws of planetary motion is that the square of a planet's orbital period around the Sun is directly proportional to the cube of its mean distance from the Sun. This relationship holds true for circular orbits as well. It is important to note that the force of gravity acts as the centripetal force for the circular motion. Therefore, the planet's acceleration is always directed towards the center of the orbit. This allows us to use the principles of circular motion to derive this relationshi:

$$\frac{GMm}{r^2} = m\frac{v^2}{r}. \tag{5.13}$$

Then replacing the speed v with $4\pi r/T$, where T is the rotational period (the time for a full orbit), we obtain:

$$T^2 = (\frac{4\pi^2}{GM})r^3. \tag{5.14}$$

The same outcome can be achieved for orbits that are elliptical, where the radius r is substituted with the semi-major axis a. The constant ratio between

T^2 and a^3 is determined by the quantity $4\pi^2/GM$, which applies to all planets orbiting the Sun. This relationship is confirmed by the data presented in Table 4.1. By measuring T and a for an orbiting body, we can calculate the mass of the central body, regardless of the orbiting body's mass. It should be noted that this method does not provide any information about the mass of the orbiting body itself.

Planet	Semi-major Axis (10^{10} m)	Period (yr)	T^2/a^3 ($10^{-34}\frac{\text{yr}^2}{\text{m}^3}$)
Mercury	5.79	0.241	2.99
Venus	10.8	0.615	3.00
Earth	15.0	1.00	2.96
Mars	22.8	1.88	2.98
Jupiter	77.8	11.9	3.01
Saturn	143	29.5	2.98
Uranus	287	84.0	2.98
Neptune	450	164.8	2.99
Pluto	591	247.7	2.99

Table 5.1: Table of Planetary Data

5.7 Velocities in Different Orbits in Celestial Mechanics

In celestial mechanics, the motion of a celestial body is often described in terms of its orbit around another celestial body. There are four types of conic section orbits: circular, elliptical, parabolic, and hyperbolic. Each orbit has a specific set of characteristics, including its velocity and energy.

Circular Orbit

A circular orbit is a special case of an elliptical orbit where the semi-major axis a is equal to the radius r. Kepler's third law states that the square of the orbital period T is proportional to the cube of the semi-major axis a:

$$T^2 = \frac{4\pi^2}{GM}a^3. \tag{5.15}$$

The velocity of a circular orbit can be derived by equating the centripetal force F_c with the gravitational force F_g:

$$F_c = F_g, \tag{5.16}$$

which can be written as:

$$\frac{mv^2}{r} = \frac{GMm}{r^2}. \tag{5.17}$$

Simplifying this expression, we obtain the velocity of a circular orbit:

$$v = \sqrt{\frac{GM}{r}}. \tag{5.18}$$

Elliptical Orbit

An elliptical orbit is described by the semi-major axis a and the eccentricity e, where e is the ratio of the distance between the foci of the ellipse to the length of the major axis. Kepler's second law states that the area swept out by the radius vector in a given time is constant, which implies that the speed of the orbiting body varies along the orbit.

We can derive the velocity of an elliptical orbit by using the conservation of angular momentum, which states that the product of the mass m, the velocity v, and the distance r from the center of mass to the orbiting body is constant:

$$mvr = h, \tag{5.19}$$

where h is the specific angular momentum. We can express h in terms of the semi-major axis a and the eccentricity e:

$$h = \sqrt{GMa(1-e^2)}. \tag{5.20}$$

Using Kepler's second law, we can express the speed v at any point in the orbit as:

$$v = \frac{h}{r}\frac{1}{1+e\cos\theta}, \tag{5.21}$$

where θ is the true anomaly, which is the angle between the position vector of the orbiting body and the pericenter of the orbit.
We can simplify this expression by expressing the distance r in terms of the semi-major axis a and the eccentricity e:

$$r = \frac{a(1-e^2)}{1+e\cos\theta}. \tag{5.22}$$

Substituting this expression into the equation for the velocity, we obtain the velocity of an elliptical orbit:

$$v = \sqrt{GM\left(\frac{2}{r}-\frac{1}{a}\right)} = \sqrt{\frac{2GM}{r}-\frac{GM}{a}}. \tag{5.23}$$

Parabolic Orbit

A parabolic orbit is an orbit in which the distance between the two bodies approaches infinity. The velocity of a parabolic orbit can be derived using the concept of specific energy, which is the sum of the kinetic and potential energy per unit mass of the orbiting body. For a parabolic orbit, the specific energy is zero, which means that the kinetic energy is equal in magnitude to the potential energy. Thus, the total energy is also zero.

Using the conservation of energy, we can equate the kinetic energy to the negative potential energy:

$$\frac{1}{2}mv^2 = -\frac{GMm}{r}. \tag{5.24}$$

Solving for the velocity, we obtain the velocity of a parabolic orbit:

$$v = \sqrt{\frac{2GM}{r}}. \tag{5.25}$$

In conclusion, the velocity of conic section orbits can be derived using the principles of classical mechanics and the laws of gravity. Circular orbits have a constant velocity, while elliptical orbits have varying speeds along the orbit. Parabolic and hyperbolic orbits have specific energies that result in unique velocities. By understanding the velocity of conic section orbits, we can better understand the motion of celestial objects and their interactions with each other.

CHAPTER 6

Practice problems

1. The luminosity of the Sun is $L = 3.85 \times 10^{26}\ W$. If the distance of the Earth from Sun is equal to 1 AU. Determine what is the flux that Earth receives from Sun.($1AU = 1.496 \times 10^{11}\ m$)
Solution The flux (or intensity) that the Earth receives from the Sun can be calculated using the inverse square law of radiation, which states that the intensity of radiation decreases with the square of the distance from the source. The flux at a distance r from a source with luminosity L is given by:

$$F = \frac{L}{4\pi r^2},$$

where F is the flux received per unit area.
Substituting the given values, we get:

$$F = \frac{3.85 \times 10^{26}\ W}{4\pi(1.496 \times 10^{11}\ m)^2}$$

Simplifying this expression, we get:

$$F = 1361\ \mathrm{W/m}^2$$

Therefore, the flux that the Earth receives from the Sun is 1361 $\mathrm{W/m}^2$. This quantity is known as the solar constant and is an important parameter in climate science and solar energy applications.

2. Determine the brightness of a star with the magnitude of 2. You can use Sun as the known star to compare. (The apparent magnitude of our Sun is -26.8)
Solution The magnitude system used to measure the brightness of stars is logarithmic, meaning that a difference of 1 magnitude corresponds to a factor of 2.512 in brightness.

the apparent magnitude of the Sun ($m_\odot = -26.8$) as a reference, we can calculate the ratio of the brightness of the star to the brightness of the Sun:

$$\frac{L_\star}{L_\odot} = 2.512^{m_\odot - m_\star},$$

where $L_\star$ and $L_\odot$ are the luminosities of the star and the Sun, respectively, and $m_\star$ is the apparent magnitude of the star.
Plugging in the values, we get:

$$\frac{L_\star}{L_\odot} = 2.512^{-26.8-2} \approx 3.98 \times 10^{-10}.$$

Therefore, the star is about 3.98×10^{-10} times as bright as the Sun.
To convert this ratio to a measure of brightness (flux), we can use the formula:

$$\frac{F_\star}{F_\odot} = \frac{L_\star}{4\pi d_\star^2 L_\odot}$$

where $F_\star$ and $F_\odot$ are the fluxes (energy per unit area per unit time) received from the star and the Sun, respectively, and $d_\star$ is the distance to the star.

that the star is at the same distance from us as the Sun ($d_\star$ = 1 AU = 1.496×10^{11} m), we can plug in the values and solve for $F_\star$:

$$\frac{F_\star}{F_\odot} = \frac{3.98 \times 10^{-10}}{4\pi(1.496 \times 10^{11})^2} \approx 4.38 \times 10^{-5}$$

Therefore, the star has a flux (brightness) of about 4.38×10^{-5} times that of the Sun at Earth's distance.

3. Determine the limiting magnitude of an 8-inch telescope.
Solution The limiting magnitude of a telescope is the faintest magnitude of a star that can be detected by the telescope. It depends on several factors, such as the aperture of the telescope, the light pollution in the observing area, and the sensitivity of the detector (e.g., the eye or a camera).
For an 8-inch telescope, the limiting magnitude can be estimated using the formula:

$$m_e - m_t = -5 \log \frac{D_t}{D_e},,$$

where m_t is the limiting magnitude, D_t is the diameter of the telescope, and $D_e = 6\ mm$ is the diameter of the pupil. This formula assumes a dark observing site with little or no light pollution.
Substituting $D = 8 \times 25\ mm$, we get:

$$m_t = 6.5 + 5 \log \left(\frac{8 \times 25}{6} \right) = 14.1$$

Therefore, the limiting magnitude of an 8-inch telescope is approximately $m_{limit} = 14.1$. This means that the telescope can detect stars with an apparent magnitude of 14.1 or brighter.

4. The **Ca , H** and **K** lines have rest wavelengths of $\lambda_{rest} = 3968.5$ Å and 3933.6 Å respectively. In the spectrum of a galaxy in the cluster *Abell* 2065 (a.k.a. the Corona Borealis Cluster), the observed wavelengths of the two lines are $\lambda_{obs} = 4255.0$ Å and 4217.6 Å respectively.

(a)What is the redshift z of the galaxy?

(b)What is the distance to the galaxy?

Solution (a) The redshift z of a galaxy can be calculated using the formula:

$$z = \frac{\lambda_{obs} - \lambda_{rest}}{\lambda_{rest}}$$

For the Ca II H line, we have:

$$z_H = \frac{\lambda_{obs,H} - \lambda_{rest,H}}{\lambda_{rest,H}} = \frac{4255.0;\text{Å} - 3968.5;\text{Å}}{3968.5;\text{Å}} = 0.072.$$

For the Ca II K line, we have:

$$z_K = \frac{\lambda_{obs,K} - \lambda_{rest,K}}{\lambda_{rest,K}} = \frac{4217.6;\text{Å} - 3933.6;\text{Å}}{3933.6;\text{Å}} = 0.072.$$

Therefore, the redshift of the galaxy is $z = 0.072$.
(b) To calculate the distance to the galaxy, we can use Hubble's law, which relates the recessional velocity of a galaxy to its distance:

$$v = H_0 d,$$

where v is the recessional velocity, d is the distance, and H_0 is the Hubble constant. The recessional velocity can be calculated from the redshift using the formula:

$$v = cz,$$

where c is the speed of light. Therefore, we have:

$$d = \frac{v}{H_0} = \frac{cz}{H_0}$$

The current value of the Hubble constant is a matter of debate and has been measured to be around 73; km/s/Mpc by some recent studies. Using this value, we can calculate the distance to the galaxy as:

$$d = \frac{cz}{H_0} = \frac{(73\text{ km/s/Mpc})(0.072)(3.086 \times 10^{19}\text{ km/Mpc})}{1\text{ s}} = 168\text{ Mpc}$$

Therefore, the distance to the galaxy is $d = 168$ Mpc.

5. We are making an observation on $1st$ day of February 2022. We observe that Mars is in opposition; at the same time, we see that Jupiter is also in western quadrature:

(a) Determine the date of the next conjunction of Mars.

(b) Determine the date of the next opposition of Jupiter.

(c) Find the distance of Mars and Jupiter on $February\ 1,\ 2022$.

(d) Determine the date of the next opposition of Mars and Jupiter.

(e) What is the angle of Earth-Mars-Jupiter on the next opposition of Jupiter?

(f) Discuss the situation when all three planets are on one side of the Sun on a line. This mean that Mars and Jupiter are going to be in opposition with Earth at the same time. When do you think this happens?

Solution (a) The next conjunction of Mars will occur when Mars, Earth, and Sun are aligned in a straight line with Mars being on the same side of the Sun as Earth. The time between two such alignments is the synodic period of Mars. The synodic period of Mars is approximately 780 days. Therefore, the next

conjunction of Mars will occur after half of the synodic period, which is 390 days from the date of the observation.
(b) The synodic period of Jupiter is the time it takes for Jupiter to return to the same relative position with respect to Earth and the Sun. Since we know that Jupiter is currently in western quadrature with Earth, we can use the synodic period to determine when the next opposition will occur.

synodic period of Jupiter is approximately 398.9 Earth days. To find the time between the current western quadrature and the next opposition, we need to find the difference between the current phase angle (i.e., the angle between the Sun-Jupiter line and the Sun-Earth line) and 180 degrees, which is the angle between the Sun-Jupiter line and the Earth-Jupiter line at opposition.

western quadrature, the elongation angle is 90 degrees. Therefore, the time between western quadrature and opposition is:

$\Delta t = \frac{\cos^{-1}(\frac{1\ AU}{5.2\ AU})}{360} \times 398.9$ days $= 87.44$ days

6. The time interval between two successive oppositions of Mars is 779.9 *days*. Calculate the semi-major axis of Mars' orbit.
Solution First, we can use the given time interval between two successive oppositions of Mars to find the synodic period P_s:

$$P_s = \frac{1}{\frac{1}{P_{Mars}} - \frac{1}{P_{Earth}}}$$

where P_{Mars} is the orbital period of Mars and P_{Earth} is the orbital period of Earth.
The orbital period of Earth is approximately 365.25 days, and we are given that the time interval between two successive oppositions of Mars is 779.9 days. Therefore, we can solve for the synodic period:

$$P_s = \frac{1}{\frac{1}{P_{Mars}} - \frac{1}{365.25}} = 779.9 \text{ days}$$

Solving for P_{Mars}:

$$\frac{1}{P_{Mars}} = \frac{1}{365.25} + \frac{1}{P_s}$$

$$P_{Mars} = \frac{1}{\frac{1}{365.25} + \frac{1}{P_s}}$$

$$P_{Mars} = \frac{1}{\frac{1}{365.25} + \frac{1}{779.9}} \approx 687 \text{ days}$$

Now, we can use the formula for semi-major axis to solve for a:

$$a = \left(\frac{GP^2}{4\pi^2}\right)^{1/3} = \left(\frac{6.674 \times 10^{-11}\ \text{m}^3/\text{kg/s}^2 \times (687 \text{ days} \times 24 \text{ hours/day} \times 3600 \text{ s/hour})^2}{4\pi^2}\right)^{1/3}$$

$$a \approx 2.28 \times 10^{11} \text{ m}$$

Therefore, the semi-major axis of Mars' orbit is approximately 2.28×10^{11} meters, which is equal to 1.52 *AU*

7. The quasar $SDSS$ $1030+0524$ produces a hydrogen emission line of wavelength $\lambda_{rest} = 121.6\ nm$. On Earth, this emission line is observed to have a wavelength of $\lambda_{obs} = 885.2\ nm$:

(a) What is the redshift of this quasar?

(b) Determine the radial velocity of the quasar.

(c) Determine the distance of this quasar.

Solution (a) Using the formula $z = \frac{\lambda_{obs}-\lambda_{rest}}{\lambda_{rest}}$, we have:

$z = \frac{885.2\ nm - 121.6;nm}{121.6\ nm} \approx 6.266$

Therefore, the redshift of this quasar is $z = 6.266$.

(b) The radial velocity of the quasar can be calculated using the formula $v = cz$, where c is the speed of light. Substituting the values, we get:

$v = (2.998 \times 10^8\ m/s) \times 6.266 \approx 1.88 \times 10^9\ m/s$

Therefore, the radial velocity of the quasar is $v \approx 1.88 \times 10^9\ m/s$.

(c) We can use the Hubble's law to estimate the distance of the quasar:

$H_0 = 73.3\ (km/s)/Mpc$

$v = H_0 \times d$

where d is the distance to the quasar. Converting v to meters per second and H_0 to meters per second per meter, we get:

$v = 1.88 \times 10^9\ m/s$

$H_0 = 73.3\ (km/s)/Mpc \approx 2.39 \times 10^{-18}\ (m/s)/m$

Substituting the values, we get:

$d = \frac{v}{H_0} \approx \frac{1.88\times10^9\ m/s}{2.39\times10^{-18}\ (m/s)/m} \approx 7.87 \times 10^{27}\ m$

Therefore, the distance to the quasar is $d \approx 7.87 \times 10^{27}\ m$.

8. Solve completely the spherical triangle ABC, and find the area of the triangles: (assume R=1)

(a) $a = 34°\ 46'$, $b = 27°\ 22'$, $C = 72°\ 31'$

(b) $b = 98°\ 18'$, $C = 24°\ 49'$, $A = 68°\ 36'$

(c) $a = 14°\ 03'$, $b = 53°\ 32'$, $c = 124°\ 14'$

(d) $A = 23°\ 32'$, $B = 102°\ 38'$, $C = 34°\ 44'$

Solution To solve a spherical triangle completely, we need to find all three sides and all three angles. We will use the following formulas for the spherical law of cosines and sines:

Law of cosines:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\cos b = \cos a \cos c + \sin a \sin c \cos B$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

Law of sines:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

The area of a spherical triangle can be found using the following formula:

$$\text{Area} = R^2 \cdot (A + B + C - \pi),$$

where A, B, and C are the angles of the triangle.

(a) $a = 34°;46';, b = 27°;22';, C = 72°;31'$ We have two sides and an angle opposite one of the sides. Using the law of cosines, we can find the third side and then use the law of sines to find the other two angles.

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

$$\cos(34°46') \cos(27°22') + \sin(34°46') \sin(27°22') \cos(72°31') \approx 0.8096$$

Therefore, $\cos^{-1}(0.8096) \approx 35°22'38''$ is the length of side c.
Using the law of sines:

$$\frac{\sin a}{\sin A} = \frac{\sin c}{\sin C} \Rightarrow \sin A = \frac{\sin a \sin C}{\sin c} \approx 0.6709$$

$$\frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \Rightarrow \sin B = \frac{\sin b \sin C}{\sin c} \approx 0.5205$$

Therefore, $\sin^{-1}(0.6709) \approx 42°47'11''$ is the measure of angle A, and $\sin^{-1}(0.5205) \approx 31°26'55''$ is the measure of angle B.
To find the area, we can use the formula:

$$\text{Area} = R^2 \cdot (A + B + C - \pi) \approx 0.0042$$

(b) $b = 98°\ 18';, C = 24°\ 49';, A = 68°\ 36'$ We have one side and two angles opposite to it. Using the law of sines, we can find the other two sides, and then use the law of cosines to find the remaining angle.
Let a be the side opposite to angle A, and c be the side opposite to angle C. Then, from the law of sines:

$$\frac{\sin a}{\sin A} = \frac{\sin c}{\sin C} \Rightarrow \sin a = \frac{\sin A \sin C}{\sin b} \approx 0.9907$$

$$\frac{\sin c}{\sin C} = \frac{\sin b}{\sin B} \Rightarrow \sin c = \frac{\sin C \sin b}{\sin B} \approx 0.6478$$

Now, using the law of cosines to find the third side:

$$\cos b = \cos a \cos c + \sin a \sin c \cos B$$

$$\Rightarrow \cos B = \frac{\cos b - \cos a \cos c}{\sin a \sin c} \approx 0.4969$$

$$\Rightarrow B = \cos^{-1}(0.4969) \approx 60°44'12''$$

Finally, to find the remaining angle:

$$A + B + C = \pi \Rightarrow A = \pi - B - C \approx 46°34'9''$$

To find the area, we can use the formula:

$$\text{Area} = R^2 \cdot (A + B + C - \pi) \approx 0.0806$$

(c) $a = 14°\ 03'\ , b = 53°\ 32'\ , c = 124°\ 14'$ We have all three sides. Using the law of cosines, we can find all three angles.

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \approx 0.1915$$

$$\cos B = \frac{\cos b - \cos a \cos c}{\sin a \sin c} \approx -0.6224$$

$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b} \approx -0.8295$$

Since all three cosines are negative, we have an obtuse triangle. Therefore, we have to use the supplementary formula for each angle:

$$A = \pi - \cos^{-1}(\cos A) \approx 154°20'35''$$

$$B = \pi - \cos^{-1}(\cos B) \approx 122°29'55''$$

$$C = \pi - \cos^{-1}(\cos C) \approx 122°15'30''$$

To find the area, we can use the formula:

$$\text{Area} = R^2 \cdot (A + B + C - \pi) \approx 0.0245$$

(d) $A = 23°\ 32'$, $B = 102°\ 38'$, $C = 34°\ 44'$ We have all three angles. Using the law of sines, we can find all three sides.
Let a be the side opposite to angle A, b be the side opposite to angle B, and c be the side opposite to angle C. Then, from the law of sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Thus, we have:

$$a = \frac{\sin A}{\sin C} c \approx 0.4501$$

$$b = \frac{\sin B}{\sin C} c \approx 1.2467$$

$$c = \frac{\sin C}{\sin C} c = R \approx 1$$

To find the area, we can use the formula:

$$\text{Area} = R^2 \cdot (\alpha + \beta + \gamma - \pi) = R^2 \cdot (A + B + C - \pi) \approx 0.2796$$

9. Two cities A and B on the same parallel of latitude $\phi = 43°\ 39'\ N$ are $127°22'$ apart in longitude. Calculate in kilometers:

(a)their distance apart along the parallel. (On the small circle between the cities with same latitude)

(b)the great circle distance AB.

(c)Determine the highest latitude of the great circle between two cities.

Solution We can use the following formulas for the calculations:
For a sphere with radius R, the length L of a small circle with radius r and central angle θ is $L = R\theta$. The great circle distance d between two points on a sphere with radius R and colatitudes ϕ_1 and ϕ_2 and difference in longitudes $\Delta\lambda$ is $d = R\cos^{-1}(\sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda)$.
(a) To calculate the distance between two points along a parallel of latitude, we can assume that the Earth is a sphere with radius $R = 6,371$ km and use the formula $L = R\theta$, where θ is the central angle between the two points on the sphere. The distance between the two cities along the parallel is

simply the arc length of the parallel between the two longitudes, which is $\theta = \frac{127°22'}{360°} \cdot 2\pi R \cos\phi \approx 14,105$ km.
(b) To calculate the great circle distance between the two cities, we can use the formula $d = R\cos^{-1}(\sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda)$, where $\phi_1 = \phi_2 = 43°39'$ N and $\Delta\lambda = 127°22'$. Then, we have:

$$d = 6,371 \text{ km} \cos^{-1}(\sin 43°39' \text{ N} \sin 43°39' \text{ N} + \cos 43°39' \text{ N} \cos 43°39' \text{ N} \cos 127°22')$$

$$d \approx 9,325 \text{ km}$$

Therefore, the great circle distance between the two cities is approximately 9,325 km.

10. An observer is tracking Rigel $\delta_R = -8°\ 12'$, $\alpha_R = 5^h\ 14^m$ in Toronto: ($\phi_{Toronto} = 43.65°\ N$)

(a) What is the maximum altitude of this star in Toronto's sky?
(b) What is the star's Azimuth at rise? What about the setting Azimuth?
(c) What is the star's Azimuth and Hour angle when its altitude is $h = 8°$?
(d) What is the star's altitude and Azimuth when $t = 1^h\ 53^m$?
(e)What angle does the star's path make with horizon at rise/set?

11. Determine the ecliptic coordinates of Rigel $\delta_R = -8°\ 12'$, $\alpha_R = 5^h\ 14^m$.

12. Show that the point on the horizon at which a star rises is:

$$sin^{-1}(\sec\phi \sin\delta)$$

Solution To solve this problem, we will need to use some basic trigonometric identities and the geometry of the celestial sphere. Let's begin by defining some terms:

ϕ is the observer's latitude, measured in degrees or radians. δ is the star's declination, measured in degrees or radians. H is the star's hour angle, measured in hours or radians. h is the altitude of the star, measured in degrees or radians. Using these definitions, we can write the following equations:

$$\sin h = \sin\phi \sin\delta + \cos\phi \cos\delta \cos H$$

$$\cos H = \frac{\sin h - \sin\phi \sin\delta}{\cos\phi \cos\delta}$$

, let's consider the point on the horizon at which the star rises. This point is defined by $h = 0$, which means that:

$$\sin\phi \sin\delta + \cos\phi \cos\delta \cos H = 0$$

for $\cos H$, we get:

$$\cos H = -\frac{\sin\phi \sin\delta}{\cos\phi \cos\delta}$$

the identity $\sec\theta = \frac{1}{\cos\theta}$, we can write this as:

$$\cos H = -\frac{\sin\phi\sin\delta}{\cos\phi\cos\delta} = -\frac{\sin\phi}{\cos\phi}\cdot\frac{\sin\delta}{\cos\delta} = -\tan\phi\tan\delta$$

the inverse sine of both sides, we get:

$$H = \sin^{-1}(-\tan\phi\tan\delta)$$

we know that the star rises at the point on the horizon where $H = 0$, so we can set $H = 0$ and solve for the latitude of the observer:

$$0 = \sin^{-1}(-\tan\phi\tan\delta)$$
$$\tan\phi\tan\delta = 0$$
$$\tan\phi = 0 \quad \text{or} \quad \tan\delta = 0$$

The first case, $\tan\phi = 0$, corresponds to an observer at the equator ($\phi = 0$) and is not interesting for this problem. The second case, $\tan\delta = 0$, corresponds to a star at the celestial equator ($\delta = 0$) and is also not interesting for this problem. Therefore, we can assume that $\tan\phi \neq 0$ and $\tan\delta \neq 0$, which allows us to write:

$$\sin^{-1}(-\tan\phi\tan\delta) = \sin^{-1}\left(-\frac{\sin\phi\sin\delta}{\cos\phi\cos\delta}\right) = \sin^{-1}(\sec\phi\sin\delta)$$

is the desired result, which shows that the point on the horizon at which a star rises is given by $\sin^{-1}(\sec\phi\sin\delta)$.

13. We have the coordinates of Vega $\delta_V = 38^\circ\ 47'$, $\alpha_V = 18^h\ 36^m$. A person in Toronto ($\phi_{Toronto} = 43.65^\circ\ N$) is observing this star:
(a) Determine the hour angle of Vega when it rise/set.
(b) What is the Azimuth of rise and set of Vega in Toronto's horizon?
(c) Determine its maximum altitude in Toronto.
(d) Determine the total time Vega is above horizon.
(e) On which date does Vega rise at the same time as the Sun in Toronto?

14. The parallax angle of a star is measured to be 0.4 arcseconds. What is the distance to the star in parsecs? Assume that the distance to the star is much greater than the radius of the Earth's orbit.
Solution The parallax angle θ is related to the distance d to the star by the formula:
$\theta = \frac{1\ \mathrm{AU}}{d}$
where 1 AU is the mean distance between the Earth and the Sun, which is approximately 1.496×10^{11} meters.
Converting the parallax angle to radians:
$\theta = 0.4$ arcseconds $\times \frac{\pi}{180\times3600}$ radians per arcsecond $= 1.184 \times 10^{-6}$ radians
Substituting into the formula, we get:
$d = \frac{1\ \mathrm{AU}}{\theta} = \frac{1.496\times10^{11}\ \mathrm{m}}{1.184\times10^{-6}} = \boxed{1.262 \text{ parsecs}}$
Note that the assumption that the distance to the star is much greater than the radius of the Earth's orbit is necessary to ensure that the parallax angle is small enough to measure accurately.

15. Star A has an apparent magnitude of 3.5 and a parallax of 0.05 arcseconds. Star B has an apparent magnitude of 2.0 and a parallax of 0.02 arcseconds. Which star is closer to Earth, and by how much in parsecs?
Solution We can use the formula for converting parallax to distance:
$d = \frac{1}{p}$,
where d is the distance in parsecs and p is the parallax in arcseconds.
For Star A, $p = 0.05$ arcseconds, so $d_A = \frac{1}{0.05} = 20$ parsecs.
For Star B, $p = 0.02$ arcseconds, so $d_B = \frac{1}{0.02} = 50$ parsecs.
Since Star A is farther away from Earth than Star B, it must be dimmer due to the inverse square law of light. We can use the formula:
$m_2 - m_1 = -2.5 \log_{10}\left(\frac{F_2}{F_1}\right)$,
where m_1 and m_2 are the apparent magnitudes of Star A and Star B, respectively, and F_1 and F_2 are their corresponding fluxes. We can assume that their intrinsic brightnesses are equal, so their flux ratio is simply the ratio of their distances squared:
$\frac{F_2}{F_1} = \left(\frac{d_1}{d_2}\right)^2$.
Plugging in the values, we get:
$m_2 - m_1 = -2.5 \log_{10}\left(\frac{d_1^2}{d_2^2}\right)$,
$m_2 - m_1 = -2.5 \log_{10}\left(\frac{20^2}{50^2}\right) = -1.95$.
Therefore, Star A is dimmer than Star B by 1.95 magnitudes.

16. A star has an apparent magnitude of $m_v = 3.5$ and a parallax of $0.03''$.
(a) Calculate the star's distance from Earth in parsecs.
(b) Calculate the star's absolute magnitude.
(c) If the star has a luminosity of $10^2 L_\odot$, what is its radius in units of the Sun's radius?
Solution To solve this problem, we will use the formula:
$m_v - M_v = 5 \log \frac{d}{10}$
where m_v is the apparent magnitude, M_v is the absolute magnitude, d is the distance in parsecs, and $L_\odot$ is the luminosity of the Sun.
We will also use the formula:
$L = 4\pi R^2 \sigma T^4$
where L is the luminosity, R is the radius, σ is the Stefan-Boltzmann constant, and T is the effective temperature. We can assume that the effective temperature of the star is similar to that of the Sun, $T = 5778$ K.
(a) The distance d in parsecs is given by:
$d = \frac{1}{p}$
where p is the parallax in arcseconds. Therefore, we have:
$d = \frac{1}{0.03} = 33.33$ pc
(b) To find the absolute magnitude M_v, we can rearrange the formula:
$M_v = m_v - 5 \log \frac{d}{10}$ Substituting in the values we have:
$M_v = 3.5 - 5 \log \frac{33.33}{10} = -0.32$
(c) To find the radius R of the star in units of the Sun's radius, we can use the formula for luminosity:
$L = 4\pi R^2 \sigma T^4$
Rearranging for R, we have:
$R = \sqrt{\frac{L}{4\pi\sigma T^4}}$

Substituting in the values we have:

$R = \sqrt{\frac{10^2 L_\odot}{4\pi\sigma(5778\text{ K})^4}} = 6.48 R_\odot$

Therefore, the star has a radius of 6.48 times the radius of the Sun.

17. A star has an apparent magnitude of $m_v = 2.5$ and a parallax of $p = 0.05$ arcseconds. Assuming that the star's absolute magnitude is $M_v = 0.5$, calculate:

(a) The distance to the star in parsecs.

(b) The luminosity of the star in solar luminosities.

Solution (a) The distance to the star can be calculated using the formula:

$d = \frac{1}{p}$

Substituting $p = 0.05$ arcseconds, we get:

$d = \frac{1}{0.05\text{ arcsec}} = 20\text{ pc}$

Therefore, the distance to the star is 20 parsecs.

(b) The luminosity of the star can be calculated using the formula:

$L = 10^{0.4(M_\odot - M_v)}, L_\odot$

where $M_\odot = 4.83$ is the absolute magnitude of the Sun, and $L_\odot = 3.828 \times 10^{26}$ W is the luminosity of the Sun.

Substituting $M_v = 0.5$ and simplifying, we get:

$L = 10^{0.4(4.83-0.5)}, L_\odot = 100.3, L_\odot$

Therefore, the luminosity of the star is 100.3 times the luminosity of the Sun.

18. A telescope has a focal length of 1000mm and an eyepiece with a focal length of 20mm. What is the magnification of the telescope? If the telescope is used to observe an object at a distance of 2000m, what is the angular magnification?

Solution The magnification of the telescope is given by:

$M = \frac{f_{obj}}{f_{eyepiece}}$

where f_{obj} is the focal length of the objective lens/mirror, and $f_{eyepiece}$ is the focal length of the eyepiece.

Substituting the given values, we get:

$M = \frac{1000}{20} = 50$

Therefore, the magnification of the telescope is 50.

The angular magnification is given by:

$M_{ang} = M \times \frac{\theta_{obj}}{\theta_{eyepiece}}$

where θ_{obj} is the angular size of the object as seen through the objective lens/mirror, and $\theta_{eyepiece}$ is the angular size of the image formed by the eyepiece.

The angular size of an object is given by:

$\theta = \frac{d}{D}$

where d is the actual size of the object, and D is the distance from the observer to the object.

Substituting the given values, we get:

$\theta_{obj} = \frac{0.01}{2000} = 5 \times 10^{-6}$ radians

The angular size of the image formed by the eyepiece is given by:

$\theta_{eyepiece} = \frac{d_{eyepiece}}{f_{eyepiece}}$

where $d_{eyepiece}$ is the distance between the eyepiece and the image formed by the objective lens/mirror.

Assuming that the telescope is in normal adjustment, we have:

$d_{eyepiece} \approx f_{obj}$

Substituting the given values, we get:

$\theta_{eyepiece} = \frac{25\times10^{-3}}{20} = 1.25 \times 10^{-3}$ radians

Substituting these values into the formula for angular magnification, we get:

$M_{ang} = 50 \times \frac{5\times10^{-6}}{1.25\times10^{-3}} \approx 0.2$

Therefore, the angular magnification of the telescope when observing an object at a distance of 2000m is approximately 0.2 radians.

19. A telescope has a focal length of 2000 mm and a plate scale of 1 arcsecond per pixel. If we want to image the planet Jupiter, which has an angular size of approximately 50 arcseconds, what should be the size of the imaging sensor to capture the entire planet? What is the approximate pixel resolution of the image? Assume that the telescope is diffraction limited.

Solution The plate scale of a telescope is defined as the angular size of one pixel in arcseconds per millimeter. In other words:

$\text{Plate scale} = \frac{206{,}265}{\text{focal length}\times\text{pixel size}}$

where the focal length is in millimeters and the pixel size is in millimeters.

We can rearrange this equation to solve for the pixel size:

$\text{Pixel size} = \frac{206{,}265}{\text{focal length}\times\text{plate scale}}$

Plugging in the given values, we get:

$\text{Pixel size} = \frac{206{,}265}{2000\ \text{mm}\times1\ \text{arcsecond/pixel}} \approx 0.103\ \text{mm/pixel}$

To capture the entire planet Jupiter, we need an imaging sensor that is at least 50 arcseconds across. Since we know the plate scale and pixel size, we can calculate the required number of pixels:

$\text{Number of pixels} = \frac{\text{Angular size}}{\text{Plate scale}\times\text{Pixel size}}$

Plugging in the values, we get:

$\text{Number of pixels} = \frac{50\ \text{arcseconds}}{1\ \text{arcsecond/pixel}\times0.103\ \text{mm/pixel}} \approx 485\ \text{pixels}$

Note that this is only an approximate value, and the actual number of pixels needed may be higher due to other factors such as image cropping and interpolation.

Finally, we can calculate the approximate pixel resolution of the image by dividing the angular size of one pixel by the distance to Jupiter:

$\text{Pixel resolution} = \frac{\text{Plate scale}}{\text{Distance to Jupiter}}$

Since Jupiter is about 5.2 astronomical units (AU) from Earth on average, we have:

$\text{Pixel resolution} = \frac{1\ \text{arcsecond/pixel}}{5.2\ \text{AU}} \approx 0.01\ \text{arcseconds/pixel}$

This means that features on Jupiter that are at least 0.01 arcseconds in size can be resolved by the telescope.

20. A telescope with a focal length of 1000 mm is used to observe Jupiter. The angular diameter of Jupiter is 40 arcseconds.

a) What is the plate scale of the telescope in arcseconds/mm?

b) If the telescope is equipped with an eyepiece with a magnification of 100x, what is the apparent diameter of Jupiter in the eyepiece?

c) If Jupiter has an average distance from the Sun of 5.2 astronomical units (AU), what is its actual diameter in kilometers?

Solution a) The plate scale of a telescope is the angular size of an object in the sky per unit distance on the detector or the focal plane. It is given by the formula:

$$\text{Plate Scale} = \frac{\text{Angular Diameter of Object}}{\text{Focal Length of Telescope}}$$

Substituting the given values, we get:

$$\text{Plate Scale} = \frac{40''}{1000 \text{ mm}} = 0.04''/\text{mm}$$

b) The apparent diameter of Jupiter in the eyepiece is given by the formula:

$$\text{Apparent Diameter in Eyepiece} = \text{Angular Magnification} \times \text{Angular Diameter of Object}$$

The angular magnification of a telescope is given by the formula:

$$\text{Angular Magnification} = \frac{\text{Focal Length of Objective}}{\text{Focal Length of Eyepiece}}$$

Since the focal length of the objective is 1000 mm and the magnification of the eyepiece is 100x, the focal length of the eyepiece is:

$$\text{Focal Length of Eyepiece} = \frac{\text{Focal Length of Objective}}{\text{Magnification}} = \frac{1000 \text{ mm}}{100} = 10 \text{ mm}$$

Substituting the values, we get:

$$\text{Apparent Diameter in Eyepiece} = 100 \times 40'' = 4000''$$

c) The actual diameter of Jupiter can be calculated using the formula:

$$\text{Actual Diameter} = \frac{\text{Apparent Diameter}}{\text{Plate Scale} \times \text{Distance}}$$

The distance to Jupiter can be calculated using its average distance from the Sun, which is 5.2 AU. One AU is defined as the mean distance between the Earth and the Sun, which is approximately 149.6 million kilometers. Therefore, the distance to Jupiter is:

$$\text{Distance} = 5.2 \text{ AU} \times 149.6 \text{ million km/AU} = 778.72 \text{ million km}$$

Substituting the given values, we get:

$$\text{Actual Diameter} = \frac{4000''}{0.04''/\text{mm} \times 1000 \text{ mm/km} \times 778.72 \text{ million km}} \approx 139,822 \text{ km}$$

Therefore, the actual diameter of Jupiter is approximately $139,822\ km$.

21. A telescope with a focal length of 1200 mm and a plate scale of 0.5 arcseconds/mm is used to observe the full moon. The field of view of the telescope is 30 arcminutes. The magnitude of the moon is -12.7.

a) What is the apparent size of the full moon in the telescope's field of view?

b) How many pixels will the full moon occupy in an image taken with a camera that has a pixel size of 5 microns?

Solution a) The plate scale of the telescope is given by:

$$PlateScale = Angular\ Diameter/Focal\ Length$$

where the angular diameter of an object is the angle it subtends at the observer's eye. The angular diameter of the full moon is approximately 0.5 degrees or 30 arcminutes. Therefore, the plate scale of the telescope is:

$$PlateScale = 30\ arcmin/1200\ mm = 0.025\ arcmin/mm$$

The apparent size of the full moon in the telescope's field of view is given by:

$$Apparent\ Size = Field\ of\ View/Plate\ Scale$$

Substituting the given values, we get:

$$Apparent\ Size = 30\ arcmin/0.025\ arcmin/mm = 1200\ mm$$

Therefore, the apparent size of the full moon in the telescope's field of view is 1200 mm.

) The number of pixels that the full moon will occupy in an image taken with a camera that has a pixel size of 5 microns is given by:

$$Number\ of\ Pixels = (Apparent\ Size/Pixel\ Size)^2$$

Substituting the values, we get:

$$Number\ of\ Pixels = (1200\ mm/5\ microns)^2 = 5.76 \times 10^{13}\ pixels$$

Therefore, the full moon will occupy approximately 5.76×10^{13} pixels in the image.

22. A small asteroid is in an elliptical orbit around the Sun. The asteroid has a semimajor axis of 2.5 AU and an eccentricity of 0.3.

a) What is the period of the asteroid's orbit?
b) What is the asteroid's speed when it is closest to the Sun (at perihelion)?
c) What is the asteroid's speed when it is farthest from the Sun (at aphelion)?
d) What is the asteroid's orbital energy?

Solution a) The period T of an object in an elliptical orbit can be calculated using Kepler's third law:

$$T^2 = \frac{4\pi^2}{GM}a^3$$

where G is the gravitational constant, M is the mass of the central object (in this case, the Sun), and a is the semimajor axis of the orbit. Substituting the given values, we get:

$$T^2 = \frac{4\pi^2}{(6.674 \times 10^{-11}\ \mathrm{m^3/kg\ s^2})(1.989 \times 10^{30}\ \mathrm{kg})}(2.5\ \mathrm{AU})^3$$

Converting AU to meters, and taking the square root, we get:

$$T = 4.33\ \text{years}$$

Therefore, the period of the asteroid's orbit is approximately 4.33 years.

) At perihelion, the asteroid is closest to the Sun, and therefore its speed is at a maximum. The speed of an object in an elliptical orbit can be calculated using the following equation:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

where r is the distance between the asteroid and the Sun, and a is the semimajor axis of the orbit. At perihelion, the distance r is equal to $(1-e)a$, where e is the eccentricity of the orbit. Substituting the given values, we get:

$$v_{\text{peri}} = \sqrt{(6.674 \times 10^{-11}\ \text{m}^3/\text{kg s}^2)(1.989 \times 10^{30}\ \text{kg})\left(\frac{2}{(1-0.3)(2.5\ \text{AU})} - \frac{1}{2.5\ \text{AU}}\right)}$$

Converting AU to meters, we get:

$$v_{\text{peri}} = 41.1\ \text{km/s}$$

Therefore, the asteroid's speed at perihelion is approximately 41.1 km/s.

) At aphelion, the asteroid is farthest from the Sun, and therefore its speed is at a minimum. Using the same equation as in part (b), we can calculate the speed of the asteroid at aphelion:

$$v_{\text{apo}} = \sqrt{(6.674 \times 10^{-11}\ \text{m}^3/\text{kg s}^2)(1.989 \times 10^{30}\ \text{kg})\left(\frac{2}{(1+0.3)(2.5\ \text{AU})} - \frac{1}{2.5\ \text{AU}}\right)}$$

Converting AU to meters, we get:

$$v_{\text{apo}} = 24.4\ \text{km/s}$$

Therefore, the asteroid's speed at aphelion is approximately 24.4 km/s.

) The orbital energy of an object in an elliptical orbit can be calculated using the following equation:

$$E = -\frac{GMm}{2a}$$

where m is the mass of the asteroid. Substituting the given values, we get:

$$E = -\frac{(6.674 \times 10^{-11}\ \text{m}^3/\text{kg s}^2)(1.989 \times 10^{30}\ \text{kg})(m)}{2(2.5\ \text{AU})(1.496 \times 10^{11}\ \text{m/AU})}$$

Converting AU to meters, and substituting m with the mass of the asteroid, we get:

$$E = -\frac{(6.674 \times 10^{-11}\ \text{m}^3/\text{kg s}^2)(1.989 \times 10^{30}\ \text{kg})(5 \times 10^{10}\ \text{kg})}{2(2.5\ \text{AU})(1.496 \times 10^{11}\ \text{m/AU})}$$

Simplifying, we get:

$$E = -6.35 \times 10^{22}\ \text{J}$$

Therefore, the asteroid's orbital energy is approximately -6.35×10^{22} J.

Appendices

APPENDIX A

Math Appendix

In this appendix, we discuss second-degree equations such as:

$$x^2 + y^2 = 1 \quad y = x^2 + 1 \quad \frac{x^2}{9} + \frac{y^2}{4} = 1 \quad x^2 - y^2 = 1$$

which represents a circle, a parabola, an ellipse, and a hyperbola, respectively. The graph of such an equation in x and y is the set of all points (x, y) that satisfy the equation; it gives a visual representation of the equation. Conversely, given a curve in the xy-plane, we may have to find an equation that represents it, that is, an equation satisfied by the coordinates of the points on the curve and by no other point. This is the other half of the basic principle of analytic geometry formulated by Descartes and Fermat. The idea is that if an algebraic equation can represent a geometric curve, then the rules of algebra can be used to analyze the geometric problem.

A.1 Circles

As an example of this type of problem, let's find an equation of the circle with radius r and center (h, k). By definition, the circle is the set of all points $P(x, y)$ whose distance from the center $C(h, k)$ is r. (See Figure A1.)

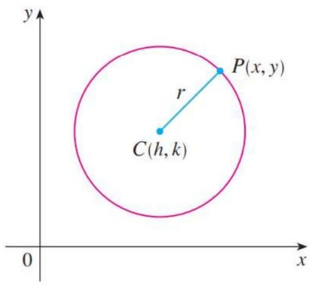


Figure A.1: Sample circle

Thus P are on the circle if and only if $|PC| = r$. From the distance formula, we have:

$$\sqrt{(x-h)^2 + (y-k)^2} = r,$$

or equivalently, squaring both sides, we get

$$(x-h)^2 + (y-k)^2 = r^2,$$

This is the desired equation.

Equation of a circle: An equation of the circle with center (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2,$$

In particular, if the center is the origin $(0, 0)$, the equation is

$$x^2 + y^2 = r^2.$$

A.2 Parabolas

We regard a parabola as a graph of an equation of the form $y = ax^2 + bx + c$. Let's draw the graph of the parabola $y = x^2$. We set up a table of values, plot points, and join them by a smooth curve to obtain the graph in Figure A2.

x	$y = x^2$
0	0
$\pm\frac{1}{2}$	$\frac{1}{4}$
± 1	1
± 2	4
± 3	9

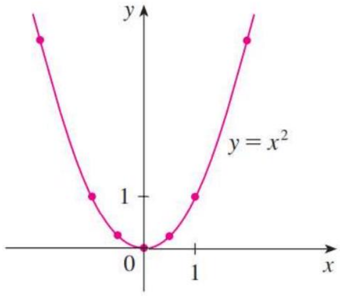


Figure A.2: Sample parabola

Figure A.3 shows the graphs of several parabolas with equations of the form $y = ax^2$ for various values of the number a. In each case the vertex, the point where the parabola changes direction, is the origin. We see that the parabola $y = ax^2$ opens upward if $a > 0$ and downward if $a < 0$ (as in Figure A.3).
Notice that if (x, y) satisfies $y = ax^2$, then so does $(-x, y)$. This corresponds to the geometric fact that if the right half of the graph is reflected about the y-axis, then the left half of the graph is obtained. We say that the graph is **symmetric with respect to the y-axis**.
The graph of an equation is symmetric with respect to the y-axis if the equation is unchanged when x is replaced by $-x$.

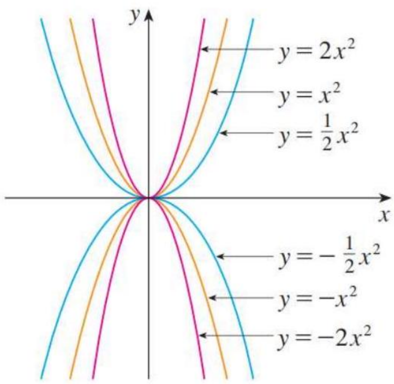


Figure A.3: Graphs of several parabolas with different a values

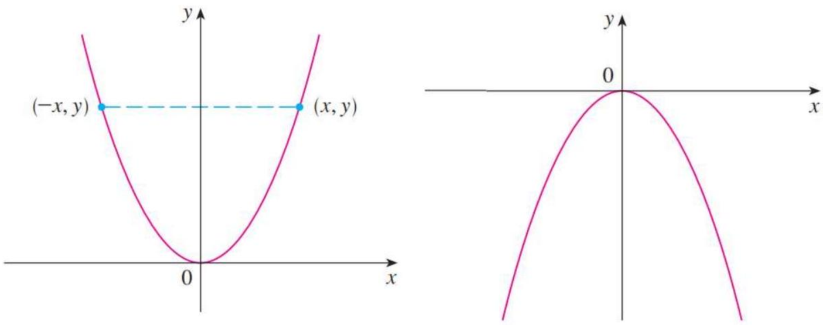


Figure A.4: different parabolas

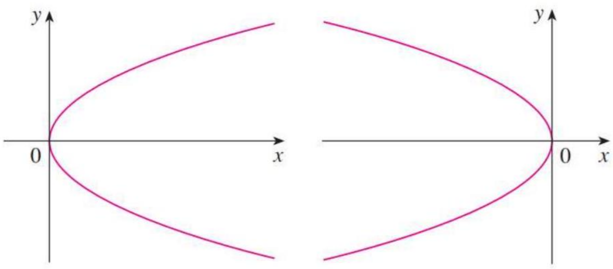


Figure A.5: different parabolas

If we interchange x and y in the equation $y = ax^2$, the result is $x = ay^2$, which also represents a parabola. (Interchanging x and y amounts to reflecting about the diagonal line $y = x$.) The parabola $x = ay^2$ opens to the right if $a > 0$ and to the left if $a < 0$. (See Figure A.5.) This time the parabola is symmetric with respect to the x-axis because if (x, y) satisfies $x = ay^2$, then so does $(x, -y)$.

The graph of an equation is symmetric with respect to the x-axis if the equation is unchanged when y is replaced by $-y$.

A.3 Ellipses

The curve with equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are positive numbers, is called an **ellipse** in standard position. Equation above is unchanged if x is replaced by $-x$ or y is replaced by $-y$, so the ellipse is symmetric with respect to both axes. As a further aid to sketching the ellipse, we find its intercepts.
The **x-intercepts** of a graph are the x-coordinates of the points where the graph intersects the x-axis. They are found by setting $y = 0$ in the equation of the graph.
The **y-intercepts** are the y-coordinates of the points where the graph intersects the y-axis. They are found by setting $x = 0$ in its equation.
If we set $y = 0$ in the equation of ellipse, we get $x^2 = a^2$ and so the x-intercepts are $\pm a$. Setting $x = 0$, we get $y^2 = b^2$, so the y-intercepts are $\pm b$. Using this information, together with symmetry, we sketch the ellipse in Figure A.6 . If $a = b$, the ellipse is a circle with radius a.

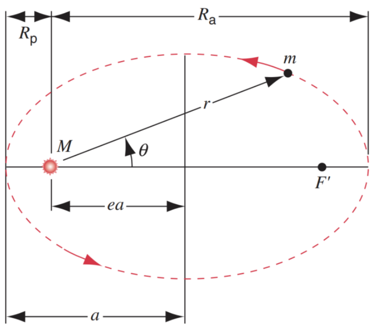


Figure A.6: Ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

A.4 Hyperbolas

The curve with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is called a hyperbola in standard position. Again, Equation above is unchanged when x is replaced by $-x$ or y is replaced by $-y$, so the hyperbola is symmetric with respect to both axes. To find the x-intercepts we set $y = 0$ and obtain $x^2 = a^2$ and $x = \pm a$. However, if we put $x = 0$ in above equation, we get $y^2 = -b^2$, which is impossible, so there is no y-intercept. In fact, we obtain:

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geqslant 1$$

which shows that $x^2 \geqslant a^2$ and so $|x| = \sqrt{x^2} \geqslant a$. Therefore we have $x \geqslant a$ or $x \leqslant -a$. This means that the hyperbola consists of two parts, called its **branches**. It is sketched in Figure A.7.

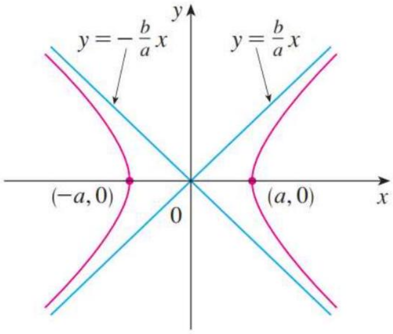


Figure A.7: Hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

In drawing a hyperbola it is useful to draw first its **asymptotes**, which are the lines $y = (b/a)x$ and $y = -(b/a)x$ shown in Figure A.7. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes.
By interchanging the roles of x and y we get an equation of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

which also represents a hyperbola and is sketched in Figure A.8:

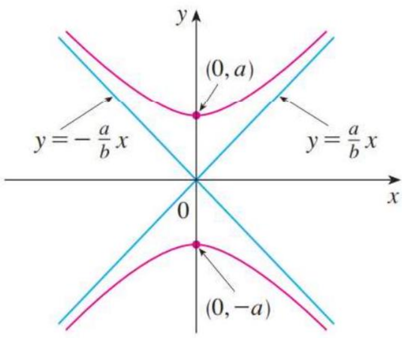


Figure A.8: Hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

A.5 Angles

Angles can be measured in degrees or in radians (abbreviated as rad). The angle given by a complete revolution contains 360°, which is the same as 2π rad. Therefore:

$$\pi \,\text{rad} = 180°$$

and

$$1\text{rad} = \left(\frac{180}{\pi}\right)^{\circ} \approx 57.3° \quad 1° = \frac{\pi}{180}\text{rad} \approx 0.017\text{rad}$$

Example 1
(a) Find the radian measure of 60°. (b) Express $5\pi/4$ rad in degrees.
Solution
(a) From Equation 1 or 2 we see that to convert from degrees to radians we multiply by $\pi/180$. Therefore

$$60° = 60\left(\frac{\pi}{180}\right) = \frac{\pi}{3}\text{rad}$$

(b) To convert from radians to degrees we multiply by $180/\pi$. Thus

$$\frac{5\pi}{4}\text{rad} = \frac{5\pi}{4}\left(\frac{180}{\pi}\right) = 225°$$

In calculus we use radians to measure angles except when otherwise indicated. The following table gives the correspondence between degree and radian measures of some common angles.

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

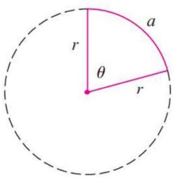


Figure A.9: Sector of a circle with central angle θ

Figure A.9 shows a sector of a circle with central angle θ and radius r subtending an arc with length a. Since the length of the arc is proportional to the size of the angle, and since the entire circle has circumference $2\pi r$ and central angle 2π, we have:

$$\frac{\theta}{2\pi} = \frac{a}{2\pi r}$$

Solving this equation for θ and for a, we obtain:

$$\theta = \frac{a}{r} \qquad a = r\theta$$

Remember that above equations are valid only when θ is measured in radians. In particular, putting $a = r$ in above equation, we see that an angle of 1 rad is the angle subtended at the center of a circle by an arc equal in length to the radius of the circle (see Figure A.10).

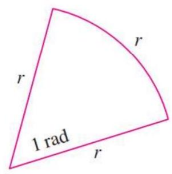


Figure A.10: Sector of a circle with its radius equal to the arc

The **standard position** of an angle occurs when we place its vertex at the origin of a coordinate system and its initial side on the positive x-axis as in Figure A.11 . A **positive** angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side. Likewise, **negative** angles are obtained by clockwise rotation as in Figure A.11.

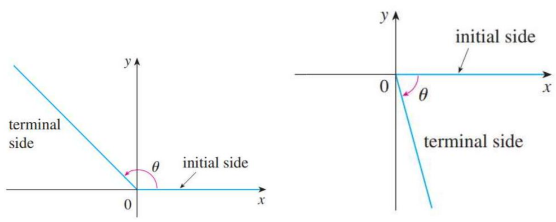


Figure A.11: Positive and negative angles

Figure A.12 shows several examples of angles in standard position. Notice that different angles can have the same terminal side. For instance, the angles $3\pi/4$, $-5\pi/4$, and $11\pi/4$ have the same initial and terminal sides because:

$$\frac{3\pi}{4} - 2\pi = -\frac{5\pi}{4} \qquad \frac{3\pi}{4} + 2\pi = \frac{11\pi}{4}$$

and 2πrad represents a complete revolution.

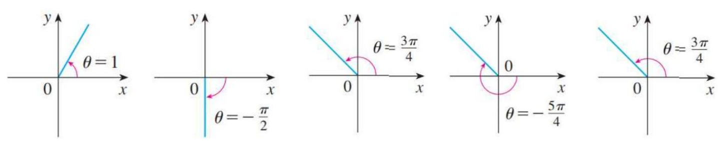


Figure A.12: Angles in standard position

A.6 Trigonometric Identities

Pythagorean Identities

$$
\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
\tan^2 x + 1 &= \sec^2 x \\
1 + \cot^2 x &= \csc^2 x
\end{aligned}
$$

Sum and Difference Formulas

$$
\begin{aligned}
\sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y \\
\cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \\
\tan(x \pm y) &= \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}
\end{aligned}
$$

Double-Angle Formulas

$$
\begin{aligned}
\sin 2x &= 2 \sin x \cos x \\
\cos 2x &= \cos^2 x - \sin^2 x \\
&= 2\cos^2 x - 1 \\
&= 1 - 2\sin^2 x \\
\tan 2x &= \frac{2 \tan x}{1 - \tan^2 x}
\end{aligned}
$$

Half-Angle Formulas

$$
\begin{aligned}
\sin \frac{x}{2} &= \pm\sqrt{\frac{1 - \cos x}{2}} \\
\cos \frac{x}{2} &= \pm\sqrt{\frac{1 + \cos x}{2}} \\
\tan \frac{x}{2} &= \frac{\sin x}{1 + \cos x}
\end{aligned}
$$

A.7 Polar Coordinates

In polar coordinates, a point in the plane is represented by an ordered pair (r, θ), where r is the distance from the origin to the point and θ is the angle between the positive x-axis and the line segment connecting the origin to the point, measured counterclockwise.

Conversion from Cartesian to Polar Coordinates

Given a point (x, y) in Cartesian coordinates, we can convert to polar coordinates as follows:

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Note that the angle θ must be adjusted to lie in the appropriate quadrant.

Conversion from Polar to Cartesian Coordinates

Given a point (r, θ) in polar coordinates, we can convert to Cartesian coordinates as follows:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Position Vector

The position vector in polar coordinates is given by

$$\mathbf{r} = r\hat{r}$$

where $\hat{r}$ is the unit vector in the radial direction, given by

$$\hat{r} = \big(\cos\theta, \sin\theta\big)$$

Velocity Vector

To derive the velocity vector in polar coordinates, we differentiate the position vector with respect to time:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$

The time derivative of the unit vector $\hat{r}$ can be found using the chain rule:

$$\frac{d\hat{r}}{dt} = \frac{d}{dt}\big(\cos\theta, \sin\theta\big) = \big(-\sin\theta, \cos\theta\big)\frac{d\theta}{dt} = \big(-\sin\theta, \cos\theta\big)\,\dot{\theta}$$

Substituting this expression into the velocity vector equation, we get

$$\mathbf{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

where $\hat{\theta}$ is the unit vector in the tangential direction, given by

$$\hat{\theta} = \big(-\sin\theta, \cos\theta\big)$$

Acceleration Vector

To derive the acceleration vector in polar coordinates, we differentiate the velocity vector with respect to time:

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = (\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$$

where $\ddot{r} = \frac{d^2r}{dt^2}$ and $\ddot{\theta} = \frac{d^2\theta}{dt^2}$ are the second derivatives of r and θ, respectively.

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