CAAO 2023

Canadian Astronomy and Astrophysics Olympiad

March 14, 2023

Full Name: ____________________________

Date of Birth: __________________________

Grade: __________________________

School: __________________________

Instructions:

- This exam comprises **8** problems.
- Both Senior and Junior students have the same exam.
- Read the instructions for each question carefully.
- Write your answers by hand, and refrain from using a computer or typing device.
- Show all your work and justify your answers in a clear and concise manner.

Note: This exam is worth **155** points.

Problem authors: V.Zaganova, Z.Li, E.Myzelev

1 Moon, planets and aircraft (15 points)

This picture captures a perfect configuration of the Moon, Venus, and Jupiter in a straight line, as captured by Picabuzz on February 27, 2023, at 19:00. Identify by photo:

(a) Is the moon currently in the waxing or waning phase?

(b) Approximate latitude of the place of observation.

(c) Angular distance from the Moon to the planets at the time of the survey.

(d) The distance of the photographer to the aircraft (aircraft fuselage length 40 meters).

(e) Altitude of the aircraft.

2 In the world of stars (12 points)

A binary star (2.28^m and 5.08^m) consists of stars that are close to each other. The plane in which the stars move is oriented along the line of sight, so that the stars periodically obscure each other. In this case, we will see the binary with three different magnitudes.

(a) Determine the brightness of a binary star in these three arrangements.

(b) What would be the brightness of the same double star, but with the plane of motion of the stars perpendicular to the line of sight?

3 Mars Mission (20 points)

A spacecraft begins its movement in an orbit around the Sun very close to the Earth, and intends to be in an orbit around the Sun that is very close to Mars. The spacecraft performs the transfer in an elliptical orbit as shown in Figure 1. The spacecraft does this with an initial impulse speed close to Earth with a magnitude of Δv_1 and then a secondary impulse speed close to Mars with a magnitude of Δv_2. Assume that these impulse speeds are applied instantaneously and ignore the change in the spacecraft's propulsion mass due to fuel consumption. Ignore the gravity of the Earth and Mars. Assume that both Earth and Mars are in circular orbits with radii R_E and $R_M = \frac{R_E}{\alpha}$ around the Sun. Consider the orbital speed of the Earth and Mars as v_E and v_M, respectively.

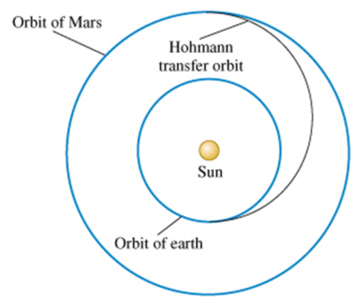


Figure 1: Hohmann transfer orbit

(a) Determine the eccentricity of the transfer orbit in terms of α.

(b) How long it takes for the spacecraft to reach Mars in years? (in terms of α)

(c) Find the necessary impulse speed required to change from a circular orbit to an elliptical orbit, i.e., Δv_1 in terms of v_E and α.

(d) Find the necessary impulse speed required to change from an elliptical orbit to a circular orbit, i.e., Δv_2 in terms of v_E and α.

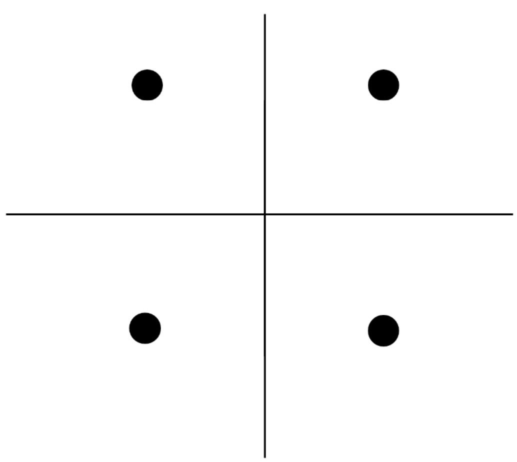
(e) If we look from the direction of the Sun, what is the separation angle between Earth and Mars at the time that the spacecraft reaches Mars? Express the answer in terms of α.

4 Binary Stars (20 points)

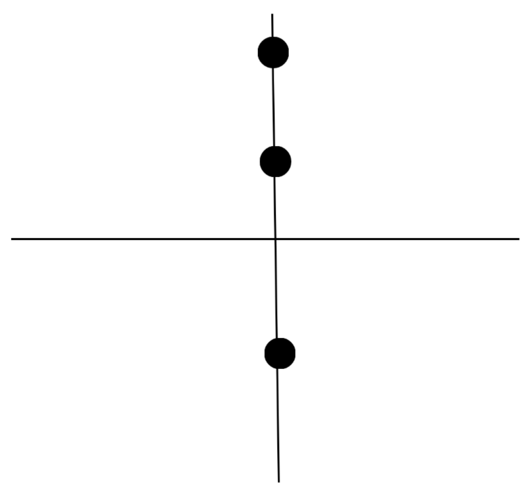
An astronomer sees 4 stars forming the vertices of a square in her telescope's FOV (Figure 1). The side length of the square is 1", and the stars are 30.1 pc away. She is so impressed that she records the coordinates of the stars in her journal. 50 years later her grandson finds her journal and points the same telescope to the same point in the night sky, relative to the other stars. He is surprised to find a completely different arrangement of stars, as pictured in Figure 2.

(a) The young astronomer suspects that the four stars are actually two binary star systems, and that the reason that there appear to be only three stars in Figure 2 is that one of the stars has been eclipsed by another. Assume that other than the orbits of the binary stars, there is no proper motion. Explain how the stars moved from the orientation in Figure 1 to Figure 2. You should state which two stars each binary system consists of, and how these stars changed their positions. Note that there is only 1 correct answer, so you should justify why the other possible pairings of binaries is impossible.

(b) Suppose that in Figure 1, the distances between the stars in both binary systems are at a maximum (both the physical distance between the stars and the angular distance as it appears to an observer on earth). Also, for simplicity assume that all orbits are circular, and that the normal vectors to the plane of rotation are either parallel to the line of sight of the observer or parallel to the vertical axis. Find the possible masses of each of the four stars in Figure 1.



(a) Figure 1

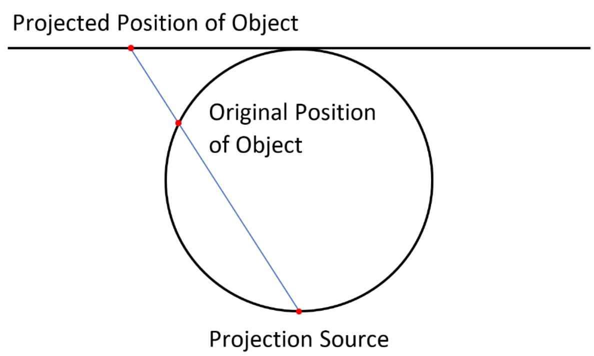


(b) Figure 2

Notes: The bottom two stars of Figure 1 and the bottom star of Figure 2 all lie on the same horizontal line. The vertical axis of each of the figures is the same. The horizontal and vertical axes bisect the sides of the square in Figure 1.

5 Stereographic Projection (20 points)

Creating a star chart requires projecting the celestial sphere onto a two-dimensional plane. This question will focus on the most common projection method, called stereographic projection. The method works by setting the celestial sphere to a fixed radius and then placing an imaginary plane on top of the sphere (some textbooks will have the plane go through the sphere's center). We imagine a light source at the bottom of the celestial sphere, and notice that every light ray emitted intersects the sphere and the plane once. We project a point on the sphere by finding the corresponding light ray and mapping the point on the sphere to the intersection of this light ray and the plane. Finally, we scale the plane to the desired size, giving us a star chart. Note that stereographic projection cannot project the entire celestial sphere as points closer and closer to the observer's nadir will become infinitely far away on the plane (to remedy this one can move the light source down past the bottom of the sphere, but then we lose some of the nice properties of the projection). Stereographic projection has the useful property that angles are preserved (this is called a conformality), as well as several other useful properties you will show below.



(a) Suppose a star has altitude h and azimuth A, and another star has altitude h' and azimuth A'. What is the distance between the projected locations of the two objects? For this and all future questions, consider the distance between the points at $h = 0°$ and $h = 90°$ (the latter being the center of the plane) to be 1 unit. Note that h can be negative.

(b) Suppose that in some planar polar coordinate system centered at the center of the plane, a star is projected onto the point (r, θ), where r is nonzero. Where will the antipodal point be projected?

(c) Suppose a small circle of radius γ on the sphere is centered at (h_0, A_0). Find the equation of the projection of this circle on the plane.

(d) Most software that generate star charts use stereographic projection. These programs typically allow the user to choose how much of the celestial sphere to project. Although it is impossible to project the entire sphere, one can project arbitrarily close to the whole celestial sphere onto a star chart. This can cause some unexpected effects, one of which is that when a circle on the sphere with its center marked is projected onto the plane, the projection of the center may not necessarily lie in the center of the projected circle. While the projected center will always lie inside the resulting circle, it may be the case that it is quite close to the circle itself.

Suppose an astronomer were to look at a space probe in a face-on circular orbit with radius $245,550\ km$ around Mars (distance from earth is $2.4555 \times 10^8\ km$). When the orbital path of the satellite is projected, it is not necessarily the case that Mars will be in the center of the circular path on the star chart. In fact, at certain locations, Mars may be so close to the space probe's orbital path that it appears to be on the path. Suppose that this astronomer wanted to play a trick on his sister, and so he sets up the software in such a way that Mars appears to be on the orbital path of the satellite. He screenshots this part of the chart, zooms in until the radius of the orbit is 1 cm, and tells his sister that the satellite is about to crash. Assuming that the human eye can distinguish distances up $1'$, and that his sister is $0.5\ m$ away from his screen, what are the possible altitudes of Mars (up to three decimal places)? You may use computer software to solve difficult trigonometric equations, but all steps must be shown.

6 Shall I compare thee to a Summer's Triangle? (18 points)

Astley "Ast" Ronomer wants to take a picture of the beautiful Summer Triangle with the smallest FOV possible while keeping all three stars visible in the image. Astley's camera may be regarded as a single ideal thin convex lens with a CCD chip located behind the lens, centered at the focus and perpendicular to the optical axis. Assume the FOV is freely adjustable by adjusting the size of the chip. (Astley has a very wide assortment of chip sizes!) The aspect ratio of the chip is fixed to 4:3, and he wishes to take the picture in landscape mode while keeping the orientation of the triangle upright" (Vega and Deneb on top, Altair on the bottom).

Astley realizes that, to minimize the FOV, all three stars of the Summer Triangle will end up on the edge of the image. So he orients his camera so that Vega and Deneb touch the upper edge of the image, whereas Altair lies on the lower edge (see figure). What is the minimal FOV that can be achieved with such a configuration? Here, the FOV is defined as the angle of the great circle arc between the midpoints of the top and bottom edges of the image.

The coordinates of the three stars of the Summer Triangle are given in the following table.

Star	α	δ
Vega	$18^h\ 36^m\ 56.3^s$	$+38°\ 47'\ 01''$
Deneb	$20^h\ 41^m\ 25.9^s$	$+45°\ 16'\ 49''$
Altair	$19^h\ 50^m\ 47.0^s$	$+08°\ 52'\ 06''$

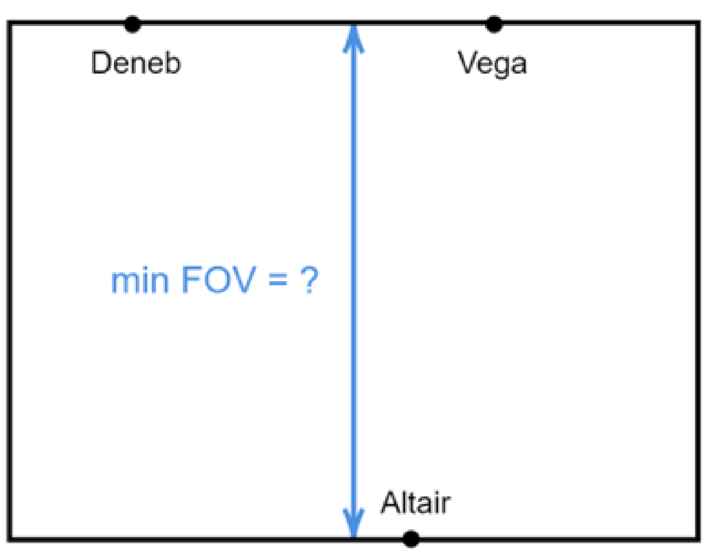


Figure 3: Astley's camera configuration for photographing the Summer Triangle.

7 Eclipsing Extrasolar Planet (25 points)

Exoplanetary systems are planetary systems outside of our solar system detected by various methods, with the transit method being a prominent one. This involves observing the light curve of a star as a planet transits across it, causing a temporary dip in brightness that provides information on the planet's size and orbital period. By studying the light curves of multiple exoplanetary systems, astronomers can gather insights into the number and size of planets, their distances and periods from their host stars, and even atmospheric composition.

The detection of exoplanetary companions through eclipsing binaries confirmed the existence of planets, while the discovery of hot Jupiters challenged the theory of planet formation. Since planets close to a star are expected to be rocky, the presence of gas giants located near their stars was unexpected. The current idea is that hot Jupiters formed at a distance from their stars and migrated inwards. Understanding this migration mechanism is currently an area of active research in the field of planet formation theory.

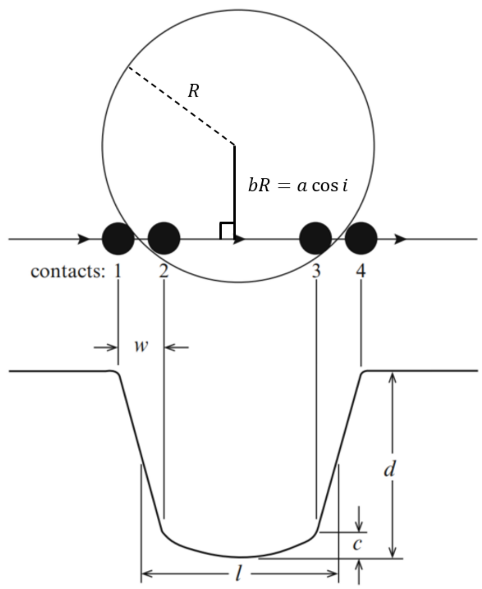


Figure 4: Schematic diagram of the eclipse

The following light curve depicts an exoplanetary system, with a circular orbit of the planet. The parameters represented in the figure are as follows: b represents the impact parameter, R represents the radius of the star, a represents the planet's orbital radius, and i represents the inclination of the planet's orbit. The task at hand is to solve for the following:

(a) Prove the transit duration equation:

$$T_{Transit} \approx \frac{PR}{\pi a}\sqrt{(1+\frac{r}{R})^2 - (\frac{a}{R}\cos i)^2}$$

Where r represents the radius of the planet, and P represents the period of the planet.

8 Annihilators (25 points)

Upset by the highly asymmetric distribution of matter vs. antimatter in our universe, you dream of living in a universe where there's an equal amount of both. The gods have been secretly listening to your wishes, and your dream comes true! They have teleported you to Annihilators', a flat universe dominated by an equal amount of annihilons and antiannihilons, in addition to radiation.

Annihilons and antiannihilons annihilate to create photons. The strength of this interaction is characterized by the coefficient α defined such that the rate of annihilation (number of annihilations per unit time per unit volume) is given by $\alpha\rho_A\rho_{\overline{A}}/m_A$, where ρ_A is the density of annihilons, $\rho_{\overline{A}}$ is the density of antiannihilons, and m_A is the mass of an (anti)annihilon.

Your goal is to measure α. As a trained cosmologist, you have made measurements on 25 galaxies and calculated the redshift z and time of emission t for each (see table below). Since Annihilators' is a flat universe and the dominant forms of matter/energy are radiation, annihilions, and antiannihilons, the Friedmann equation for Annihilators' is

$$H^2 = \frac{8}{3}\pi G(\rho_A + \rho_{\overline{A}} + \rho_r),$$

where H is the Hubble parameter and $\rho_A = \rho_{\overline{A}}$. Assume that annihilons and antiannihilons behave like dust, i.e., $p_A = p_{\overline{A}} = 0$. Radiation behaves the same as in our universe, i.e., $p_r = \frac{1}{3}\rho_r c^2$. The physical constants G and c are the same as in our universe.

Data point #	Emission time t (Gyr relative to present)	Redshift z
1	−189	7.43
2	−187	5.78
3	−187	5.78
4	−186	4.91
5	−182	3.59
6	−177	2.82
7	−175	2.45
8	−162	1.59
9	−161	1.42
10	−158	1.34
11	−155	1.39
12	−152	1.07
13	−148	1.06
14	−143	0.993
15	−131	0.783
16	−125	0.709
17	−113	0.583
18	−105	0.516
19	−92.1	0.374
20	−84.6	0.327
21	−57.0	0.197
22	−52.1	0.173
23	−37.6	0.109
24	−26.9	0.0756
25	−17.3	0.0459

Table 1: Redshift and emission time data for 25 galaxies in Annihilators'