CAAO 2025

Canadian Astronomy and Astrophysics Olympiad

March 9, 2025

Full Name: ____________________________

Date of Birth: __________________________

Grade: __________________________

School: __________________________

Instructions:

- This exam comprises **8** problems.
- Read the instructions for each question carefully.
- Please provide legible handwritten answers on separate pieces of paper.
- Show all your work and justify your answers clearly and concisely.

Note: This exam is worth **240** points.

1 Properties of Sirius and Companion (20 points)

Find the physical characteristics of the star **Sirius** (**α Canis Majoris**) and its companion based on the following observational data:

- The visible yellow stellar magnitude of Sirius is $V_1 = -1^m.46$.
- Its primary color index is $(B-V)_1 = 0^m.00$.
- The companion star has:
  - Apparent magnitude: $V_2 = +8^m.50$.
  - Color index: $(B-V)_2 = +0^m.15$.
- The **parallax** of Sirius is $\pi = 0''.375$.
- The companion revolves around Sirius with a **period** of $P = 50$ years.
- The **angular value of the semi-major axis** of the orbit is $a'' = 7''.60$.
- The ratio of the distances of both stars to the common center of mass is $a_2/a_1 = 2.3 : 1$.
- The absolute stellar magnitude of the Sun in yellow light is given as $M_{\odot V} = +4^m.77$.

2 Average Density of a B0V Star (15 points)

Calculate the average density of a main-sequence star of spectral class B0V. The star has an effective temperature of $T = 30,000$ K and a mass of approximately $15M_{\odot}$.

Use known relationships between stellar parameters to estimate its radius and subsequently determine the density. The average density ρ is given by

$$\rho = \frac{3M}{4\pi R^3},$$

where M is the mass of the star, and R is its radius, which should be determined using known relationships for star parameters.

3 Migration Paths of the Ruby-throated Hummingbird (25 points)

For centuries, humans have observed bird migrations, using them to predict seasonal changes and understand natural navigation. The **Ruby-throated Hummingbird** (*Archilochus colubris*) is one of the most remarkable migratory species, traveling thousands of kilometers each year between North and Central America. Indigenous cultures in North America regarded the return of hummingbirds as a sign of spring, while in Central America, the bird was often associated with vitality and endurance.

The **Ruby-throated Hummingbird** breeds in **Toronto** $(43.7°N, 79.4°W)$ and migrates to its wintering grounds in **San José, Costa Rica** $(9.9°N, 84.1°W)$. Due to the Coriolis effect and prevailing wind patterns, the path taken by these birds does not follow a straight line but instead forms an **elliptical trajectory**, which has been studied by ornithologists.

Through the analysis of tracking data and historical observations, scientists have identified that this migratory route can be approximated by an **ellipse**, with one of its **foci** (F_1) located in **Kansas City, USA** $(39.1°N, 94.6°W)$. However, the second focus of the elliptical migration path remains to be determined.

If the **major axis** of this elliptical migration route is 50°, answer the following:

(a) Based on the given information, determine the **coordinates of the second focus** (F_2) of this elliptical trajectory.

(b) Compute the **eccentricity** of the elliptical migration path, determine the coordinates of the **apsis points** (the closest and farthest points to the foci of the ellipse), and find the **conjugate points** along the **semi-minor axis** of the ellipse.

Note: For the purpose of this problem, assume that the Earth's surface can be approximated as a perfect sphere with a radius of 6,371 km.

4 Properties of Kuiper Belt Objects (KBOs) (30 points)

A fraction of the sunlight that is reflected by the objects is called **albedo.** The remaining light is absorbed by the body. In the context of the Solar System, Kuiper Belt Objects (KBOs) have specific albedo values that provide insights into their surface compositions and structures. The absorbed energy is then re-emitted as thermal radiation. Kuiper belt objects orbit beyond the outermost planet of the Solar System, Neptune. Therefore, they are very far from Earth, making their imaging extremely difficult. Most of these objects appear as point sources that reflect and emit radiation.

(a) An astronomer wants to measure the reflected flux from these celestial bodies. In which wavelength range should the astronomer observe, and why?

(b) Astronomer X has successfully obtained the reflected flux from these objects, denoted as F_R. This flux represents the amount of energy per unit area per unit time detected by a ground-based sensor. Suppose the observer measures F_R when the object is at opposition. Derive an expression for F_R in terms of the object's distance from the Sun, D, the Sun's luminosity, L, the object's radius, R (assuming it is spherical), and its albedo, A.

(c) Astronomer Y has measured the thermal radiation flux emitted by these objects, denoted as F_E. Derive the required expression for F_E using the relation found in part (b).

(d) In the opposition configuration, the observed KBOs have a **proper motion** $\mu = 3\ arcsecond/hr$ relative to distant stars. This motion is solely due to parallax. Using this information, estimate D and express it in astronomical units (AU).

(e) Based on the results obtained in parts (c) and (d), determine the wavelength at which astronomer Y observed the emitted radiation.

(f) Suppose both astronomers X and Y know the value of D.

  (a) Can astronomer X determine R alone?

  (b) Can astronomer Y determine R alone?

They can determine R together by combining their measurements.

5 Gamma-Ray Bursts (GRBs) in Distant Galaxies (30 points)

Gamma-ray bursts (GRBs) are among the most energetic explosions in the universe, originating from the collapse of massive stars (long GRBs) or the merger of compact objects such as neutron stars (short GRBs). These explosions occur in distant galaxies and can outshine an entire galaxy for a brief period. They typically last for a few milliseconds to several minutes and produce photons with energies around 1 MeV. Due to their immense luminosity, GRBs serve as valuable probes of the early universe and distant cosmic structures.

To observe a GRB from Earth, its received flux must not be less than 50 photons per second per square centimeter. The luminosity of these explosions is approximately 10^{49} erg/s. In the Euclidean model describing this phenomenon, it is assumed that these explosions originate from a point source and are emitted in a conical jet with an opening angle of 10°. On Earth, approximately one GRB is detected per week.

(a) What fraction of the sky can observe these GRBs?

(b) Using this information, estimate the distance to the farthest GRB that can be observed from Earth.

(c) What is the approximate occurrence frequency of GRBs in a distant galaxy?

(d) If the jet opening angle were 5° instead of 10°, how many GRBs would be observable from Earth per week?

6 Comet Ejection from the Oort Cloud (30 points)

A star with mass $0.5M_\odot$ enters the Oort Cloud with a velocity of 30 km/s. Assume the Oort Cloud is a spherical shell containing a large number of comets with a radius of 10^5 AU.

(a) At what distance (in astronomical units) must this star pass near one of the Oort Cloud comets for the comet to be ejected from the Oort Cloud? Provide any necessary assumptions for solving the problem.

(b) If we assume the age of the Solar System is 4×10^9 years, what fraction of the comets in the Oort Cloud would be ejected by this mechanism? Assume the stellar number density in the Solar neighborhood is 0.1 stars per cubic parsec, the average mass of these stars is $0.5M_\odot$, and the average velocity of these stars is also 30 km/s.

(c) If we assume the radius of the Oort Cloud is 10^4 AU, how does the fraction calculated in part (b) change?

7 Cosmological Redshift (30 points)

When an object moves away from us, the received light from that object shifts to longer wavelengths, appearing redder. This phenomenon is known as **redshift**, denoted by the quantity z. It is defined as:

$$z = \frac{\lambda_{\text{received}} - \lambda_e}{\lambda_e}$$

where $\lambda_{\text{received}}$ is the observed wavelength and λ_e is the emitted wavelength.

Edwin Hubble, by observing different galaxies, demonstrated that for distant galaxies, the relation:

$$v = H_0 d$$

holds, where $H_0 = 75$ km/s/Mpc is the **Hubble constant**, d is the distance of the galaxy from us, and v is its velocity as observed from Earth. As the distance increases, the velocity of recession increases, implying that the universe is expanding. Consequently, wavelengths of emitted light stretch, causing redshift.

(a) The scale factor of the universe at time t is given by $r(t) = a(t)r(t_0)$, where $a(t)$ is the **scale factor** and $r(t_0) = r_0$ represents the present size of the universe. Show that:

$$r = \frac{1}{1+z}$$

(b) The mass density of the universe for a universe where all particles move much slower than the speed of light (non-relativistic particles) is dependent on z. Derive the expression for this dependence.

(c) The **critical density** of the universe is defined as the density for which the total energy (sum of kinetic and potential energy) of cosmic objects is zero. Compute the total energy of cosmic objects and derive an expression for the critical density.

(d) Compute the numerical value of the critical density in units of $\text{kg/m}^{\mathbf{3}}$.

(e) Express the critical density in terms of **solar masses per cubic parsec**. How many solar masses per cubic parsec does this correspond to?

8 Satellite Motion in the ECI Frame (60 points)

The ECI (Earth-Centered Inertial) frame is an inertial reference frame centered at the Earth, where the x-axis points towards the vernal equinox, and the z-axis points towards the celestial north pole. Consequently, the xy-plane lies in the celestial equator.

It can be shown, by integrating Newton's second law in three dimensions — which are second-order differential equations — that we have 6 degrees of freedom. Therefore, to fully describe an orbit around the Earth's gravity, we need 6 parameters, known as the orbital elements, which are illustrated below in a general form.

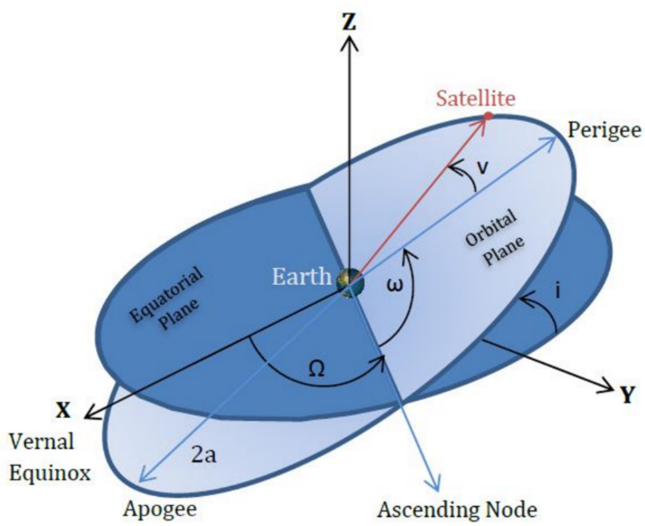

Figure 1: The satellite's orbital parameters.

Consider a satellite with the following orbital elements in the ECI frame: $i = 0$, $e = 0.15$, $a = 2.5R_e$, $\Omega = 0$, $\omega = -30°$. This means the satellite's orbit lies in the celestial equator, and the satellite follows an elliptical orbit with semi-major axis a and eccentricity e. The ellipse's major axis is rotated clockwise by 30° from the x-axis. Clearly, the Earth's center is at one of the foci of this ellipse.

In this problem, derive relationships parametrically unless a numerical value is explicitly requested by the verb "calculate."

The goal of this question is to study the motion of this satellite as observed by an observer in Toronto with coordinates: $\phi = 43.65°N$, $l = 79.5°W = -5^h18^m$. By the information gained from the British Columbia site, it is known that at 15:20 Pacific Time on December 10, the satellite is at its perigee. Since the satellite is relatively close to Earth, what the observer in Toronto sees differs slightly from what an observer at the Earth's center would see. We know that British Columbia follows Pacific Time, which is GMT -8. The Sun's orbit is assumed to be circular with a radius of 1 AU. The Sun is considered a distant object, but the satellite is not.

To better understand the concept of time in astronomy, we define the mean Sun, which moves with the same period as the real Sun but along the celestial equator instead of the ecliptic. Both start moving from the vernal equinox and meet again at the autumnal equinox. To define time on Earth, we use the mean Sun and define local time as follows:

$$LT = H_{MS} + 12^h$$

where H_{MS} is the hour angle of the mean Sun for the local observer. It is noon when this parameter is zero.

Zone time is then introduced by converting local time to Greenwich Mean Time (GMT) and adding the time zone offset. For example, Eastern Time is GMT -5:

$$ZT = GMT - 5^h$$

$$GMT = LT - l$$

(a) Calculate the declination and right ascension of the Sun on this day.

(b) From this point on, let the initial time be the moment the satellite is at its perigee. At this moment, calculate how long it has been since sunset for the observer in Toronto. Report your answer to the nearest minute.

(c) At the initial moment, provide the coordinates of the satellite and the observer in the ECI frame as 3-component vectors, both in terms of parameters and calculate numerically.

(d) Write the observer's position vector in the ECI frame as a function of time (taking the initial moment as $t = 0$).

(e) Write the satellite's position vector in the ECI frame as a function of time. (Due to the elliptical orbit, this will not be a single closed-form expression, so use Kepler's equations.)

Since the satellite's eccentricity is small, using a first-order approximation for the eccentricity, the satellite's position vector can be expressed as a time function. The first-order approximation is given by:

$$f(e) \approx f(e = 0) + f'(e = 0)e, \quad e \ll 1$$

(f) Prove that with a first-order approximation, eccentricity anomaly E can be expressed as a function of time as follows:

$$E = \Omega t - e \sin \Omega t$$

where $\Omega = \frac{2\pi}{T}$ and T is the period of the satellite.

(g) Using the first-order approximation, show that the true anomaly angle θ can be expressed as a time-dependent function:

$$\theta = E + 2e \sin E$$

$$\theta = p(t) + eq(t)$$

Describe why the function p is linear in time, and the function q is periodic with period T. Derive both as functions of the orbital parameters.

(h) Now, with the first-order expansion for eccentricity, write the satellite's position vector as a time-dependent parametric function.

(i) Formulate the condition for the satellite's rise and set as a function of the observer's and satellite's position vectors and their interactions.

(j) The previous condition results in an equation involving time. Calculate the date and time of the satellite's first rise and set as observed from Toronto in Eastern Time.

(k) Calculate the satellite's hour angle at the moments of its first rise and set(Hour angle is shown by H'):

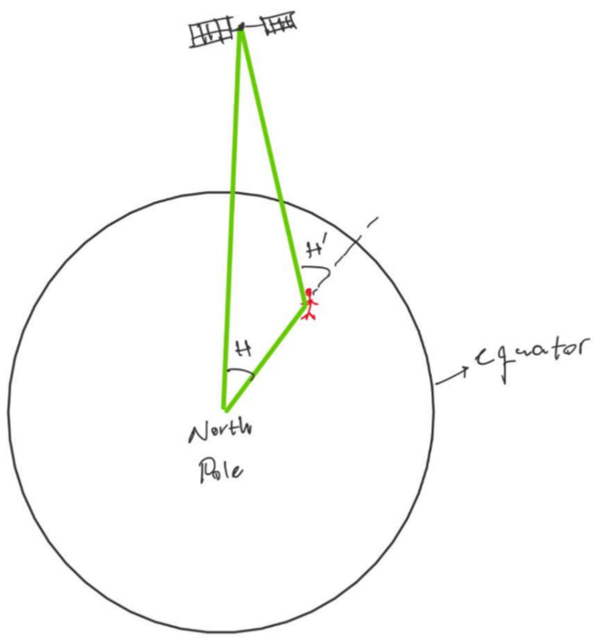

Figure 2: The satellite's hour angle is not the same for the observer and the centre of the Earth

$$H \neq H'$$

(l) The satellite's apparent motion in the sky depends on its relative tangential velocity to the observer. At the moments of the first rise and set, this tangential velocity makes an angle with the horizon, indicating how the satellite emerges from the horizon. Report this angle for both the first rise and set.

(m) The satellite's relative radial velocity to the observer causes a Doppler shift in the received signal. Find the Doppler shift as a time-dependent parametric function for the period between the satellite's rise and set. Is there a moment when the Doppler shift is zero? If so, report the date and time of this event in Eastern Time.

(n) Derive the azimuth of the satellite as a time-dependent parametric function. Is there a moment when the satellite's azimuth rate of change becomes zero and reverses direction (similar to the retrograde motion of planets)? If so, report the date and time of this event in Eastern Time.

(o) Derive the satellite's altitude as a time-dependent parametric function. At what azimuth does the satellite reach its maximum altitude? Shouldn't it be 180°? What is this maximum altitude?