Canadian Association of Amateur Astronomers (CAAA)

Canadian Astronomy and Astrophysics Olympiads

CAAO tutorial

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Preface

This guide is intended for students who wish to participate in the Canadian Astronomy and Astrophysics Olympiad (CAAO). It serves as an introductory-level introduction to the Olympiad for interested students across Canada. Each year, the highest-achieving CAAO students are selected to represent Team Canada in the International Astronomy and Astrophysics Olympiad (IOAA), and are provided with a training program to prepare them for the international competition.

While this guide contains a great deal of information, we recommend that students supplement their learning with other resources listed in the reference section. The guide includes numerous practice problems designed to complement students' learning path. Additionally, we highly encourage students to solve past CAAO problems available in a separate file.

To participate in the Canadian astronomy Olympiad, students should have a solid foundation in high school-level physics and mathematics. However, for the international Olympiads, students will need to develop an advanced understanding of physics and mathematics beyond what is typically taught in high school.

The International Astronomy and Astrophysics Olympiad (IOAA) is a prestigious international competition for high school students. Each year, the brightest students from around the world compete in this event, and Canada has been participating in the IOAA since 2013.

Resources

The Canadian Astronomy and Astrophysics Olympiad (CAAO) requires diligent preparation by interested students, and the use of appropriate resources is critical to success. Several textbooks have been identified as valuable resources in this endeavor, including:

1. Foundations of Astrophysics, authored by Barbara Ryden
2. Fundamental Astronomy, written by Karttunen et al.
3. An Introduction to Modern Astrophysics, co-authored by Bradley Carroll and Dale Ostlie
4. Astronomy Principles and Practice, by Archie E. Roy and David Clarke
5. Introduction to Cosmology, authored by Barbara Ryden

The first two textbooks are introductory level, while An Introduction to Modern Astrophysics is suitable for students with a strong background in physics and calculus. Students seeking comprehensive knowledge of spherical astronomy are advised to reference Astronomy Principles and Practice.

Additionally, students are expected to possess a solid foundation in high school-level physics and mathematics. Senior-level students are strongly encouraged to deepen their understanding of these subjects by studying calculus.

Aspiring students may benefit from exploring more advanced IOAA-level resources. The references used to compile this document have been included in the reference section for this purpose.

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PART I

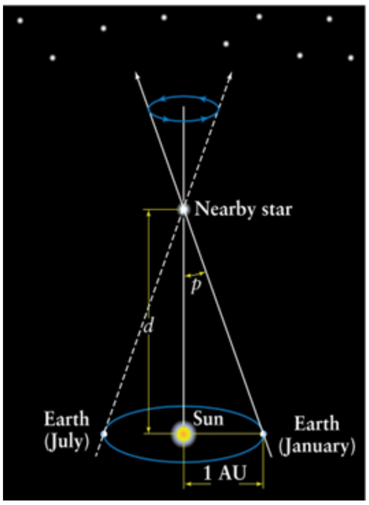
Introduction to CAAO

CHAPTER 1

Basic Concepts

1.1 Parallax

Measuring the intrinsic brightness of stars is linked with determining their distances. On Earth, the distance to the peak of a remote mountain can be determined by measuring that peak's angular position from two observation points separated by a known baseline distance. Simple trigonometry then supplies the distance to the peak. Finding the distance even to the nearest stars requires a longer baseline. As Earth orbits the Sun, two observations of the same star made 6 months apart employ a baseline equal to the diameter of Earth's orbit. These measurements reveal that a nearby star exhibits an annual back-and-forth change in its position against the stationary background of much more distant stars. a measurement of the parallax angle p (one-half of the maximum change in angular position) allows the calculation of the distance d to the star.



We can write the equation as:

$$d = \frac{1AU}{\tan p} \approx \frac{1}{p} AU \tag{1.1}$$

The angle p is smaller as the distance becomes larger. Using parallax, we are going to introduce a new distance measure, parsec.
A parsec is the distance at which 1 Astronomical Unit subtends an angle of 1 second of arc (arcsecond):

$$1 \; parsec = 3.26 \; light \; years$$

$$1pc = \frac{1AU}{(\frac{1}{3600} \times \frac{\pi}{180})} = 206265AU$$

Figure 1.1: Parallax triangle

1 arc-second is 1/60 of an arc-minute, and an arc-minute is 1/60 of one degree. Therefore, an arc-second is $1/3600 \; th$ of one degree.

1.2 Stellar Luminosity

Stars are considered as a spherical source of radiating energy due to their temperature; their total energy output can be determined by the equation below, according to their surface temperature and surface area. This total output is referred to as the stellar luminosity, L, and may be expressed as:

$$L = 4\pi R^2 \sigma T^4, \tag{1.2}$$

where R is the radius of the star, σ is known as **Stefan-Boltzmann's constant** and T is the star's surface temperature. The unit of luminosity is Watts (joules per second). For instance, the luminosity of our sun is 3.85×10^{26} W.

1.3 Brightness (radiant flux)

The brightness of a star is measured in terms of the flux received from the star. The power received per unit area at the Earth depends on the stellar luminosity and on the inverse square of the stellar distance. If the latter is known, the flux provided by the source may be readily calculated and expressed in terms of watts per square metre (W/m^2). Imagine a star of luminosity L surrounded

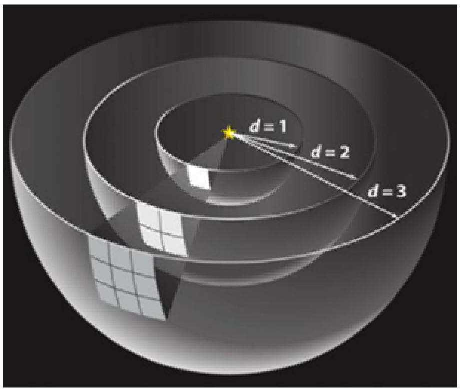


Figure 1.2: Flux vs. distance

by a huge spherical shell of radius d. Then, assuming that no light is absorbed during its journey out to the shell, the radiant flux, b, measured at distance d is related to the star's luminosity by:

$$b = \frac{L}{4\pi d^2}, \tag{1.3}$$

the denominator is simply the area of the sphere. Since L does not depend on d, the radiant flux is inversely proportional to the square of the distance from the star. This is the well-known inverse square law for light.

1.4 Magnitude system

Invented by the astronomer Hipparchus 2200 years ago, it was simply a way to "rank" the stars visible at night. The brightest were ranked as 1st magnitude, the faintest visible were ranked as 6th magnitude. In other words, the brightest stars were assigned the smallest number, the faintest the largest number. And 6 divisions were used because of the mysticism about 6, which is the first **perfect number**. The brightness ratio of rank first and sixth is 100:

$$K^5 = 100 \rightarrow K = \sqrt[5]{100} = 2.5118 \rightarrow \tfrac{b_2}{b_1} = 2.5118^{m_1 - m_2}$$

$$m_1 - m_2 = -2.5 \log \frac{b_1}{b_2}, \tag{1.4}$$

Where m is the magnitude of the stars. The magnitude system is based on the comparison; This means that you need to know the magnitude of a certain star and by comparing its brightness with other stars you can determine the magnitude of the star.

The absolute magnitude, M, is defined to be the apparent magnitude a star would have if it were located at 10 pc. Recall that a difference of 5 magnitudes between the apparent magnitudes of two stars corresponds to the smaller magnitude star being 100 times brighter than the larger-magnitude star. This allows us to find an equation for the absolute magnitude just like the apparent magnitude:

$$M_1 - M_2 = -2.5 \log \frac{L_1}{L_2}, \tag{1.5}$$

Where M is the absolute magnitude and L is the luminosity of the star. To use this equation, we need to know a specific star's absolute magnitude and luminosity to be able to compare it with other stars. We have defined two different magnitudes, absolute and apparent. We defined absolute magnitude as the apparent magnitude at a certain distance (10 parsecs), therefore, there should be a connection between the star distances and their magnitudes. This relation is called **Distance Modulus**:

$$m - M = 5 \log d - 5, \tag{1.6}$$

Where d is in parsecs, m and M are apparent and absolute magnitudes respectively. Unlike the previous magnitude relations (Equations 1.4-1.5), distance modulus is written for a single star. It is the relationship between the absolute and apparent magnitude of any star in the distance d. Using this relation, it is clear that if you have any star in the distance of 10 $parsecs$, then the two magnitudes should be equal to each other.

1.5 Limiting magnitude

Looking up at the night sky, we are not able to see all the stars; our eyes have limited light-gathering aperture (around 6mm on a dark night!). The faintest stars that your naked eye can see in the night are about $6 - 6.5$ magnitudes. However, this limiting magnitude might also change due to light pollution or atmospheric effects. For instance, in a metropolitan area, your limiting magnitude might go up to $2 - 3$ magnitudes. This would seriously limit your ability to see constellations.

We know that using optical devices would enable us to see fainter objects in the sky. For instance, if you are using a telescope since it has a bigger diameter, it is able to gather more light than your eyes can. Therefore, we have the equation below to determine the limiting magnitude of a telescope:

$$m_e - m_t = -5 \log \frac{D_t}{D_e}, \tag{1.7}$$

Where m_e is the limiting magnitude of the naked eye ($m_e \approx 6.5$), m_t is the telescope's limiting magnitude, D_e is the pupil's diameter ($D_e \approx 6\ mm$) and D_t is the telescope's diameter.

CHAPTER 2

Telescopes

2.1 Optical Telescopes

An optical telescope forms images of faint and distant stars. It can collect much more light from space than the human eye can. Optical telescopes are built in two basic designs—**refractors** and **reflectors**. The heart of a telescope is its objective, a main lens (in refractors) or a mirror (in reflectors). Its function is to gather light from a sky object and focus this light to form an image. The ability of a telescope to collect light is called its **light-gathering power**.

Light-gathering power is proportional to the area of the collecting surface, or to the square of the **aperture** (clear diameter of the main lens or mirror). The size of a telescope, such as 150 *mm* or 8 *m* (6-*inch* or 26-foot), refers to the size of its aperture. You can look at the image directly through an eyepiece, which is essentially a magnifying glass. Or you can photograph the image or record and process it electronically. Your eye lens size is about 6 *mm*. A 150 *mm* (6-*inch*) telescope has an aperture over 30 times bigger than your eye lens. Its light-gathering power is 900 times greater than that of your eye. So, a star appears over 900 times brighter with a 150 *mm* (6-*inch*) telescope than it does to your unaided eye.

Astronomers build giant telescopes to detect fainter and more distant objects. All stars appear brighter with telescopes than they do to the eye alone. The extra starlight gathered by the telescope is concentrated into a single point. Using time exposure, a giant 10 *m* (400-*inch*) telescope can image very faint stars down to about magnitude 28, which is the same apparent brightness as a candle viewed from the Moon!

2.2 Refracting Telescopes

A refracting telescope has a main, objective lens permanently mounted at the front end of a tube. Starlight enters this lens and is refracted, or bent so that it forms an image near the back of the tube. The distance from this lens to the image is its focal length. You may look at the image through a removable magnifying lens called the eyepiece. The tube keeps out scattered light, dust, and moisture. Italian astronomer Galileo Galilei (1564–1642) first pointed a refracting telescope skyward in 1609. The largest instrument he made was smaller than 50 *mm* (2-*inches*).

Today refracting telescopes range in size from a beginner's 60-*mm* (2.4-*inch*) to the largest ever built, the 1 *m* (40-*inch*) telescope at the Yerkes Observatory in Williams Bay, Wisconsin, U.S., which was completed in 1897.

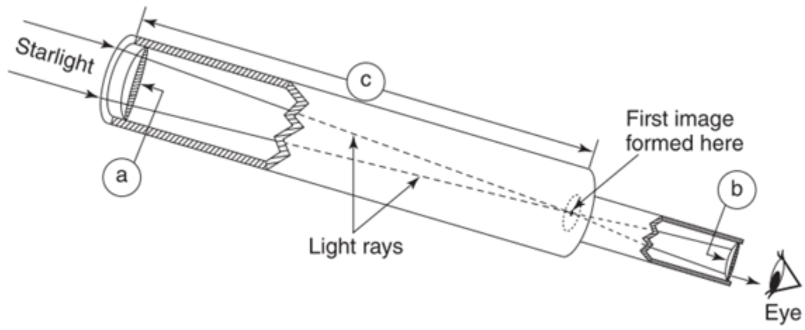


Figure 2.1: (a) Objective lens gathers the light and forms an image. (b) Eyepiece magnifies the image formed by the objective. (c) The focal length of the objective lens.

2.3 Reflecting Telescopes

A **reflecting telescope** has a highly polished curved-glass mirror, the **primary mirror**, mounted at the bottom of an open tube. When starlight shines on this mirror, it is reflected back up the tube to form an image at the **prime focus**.

You can record the image at the prime focus, or you can use additional mirrors to reflect the light to another spot. The **Newtonian telescope**, originated by British scientist Sir Isaac Newton in 1668, uses a small, flat mirror to reflect the light through the side of the tube to an eyepiece (Figure below). The **Cassegrain telescope** uses a small convex mirror, a *secondary mirror*, to reflect the light back through a hole cut in the primary mirror at the bottom end of the tube. It is more compact than a refractor or Newtonian reflector of the same aperture. The **Schmidt-Cassegrain** telescope combines an extremely short-focus spherical primary mirror at the back end of a sealed tube with a thin lens at the front.

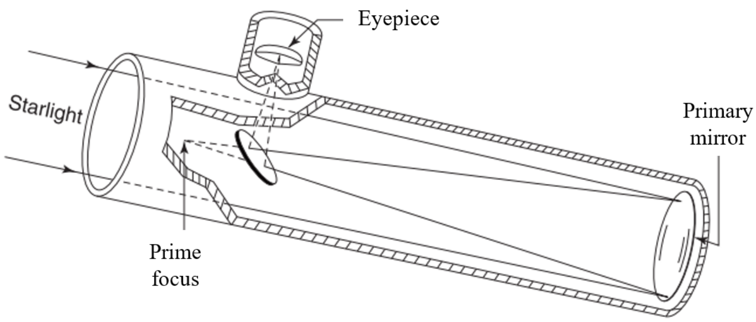


Figure 2.2: Newtonian reflecting telescope

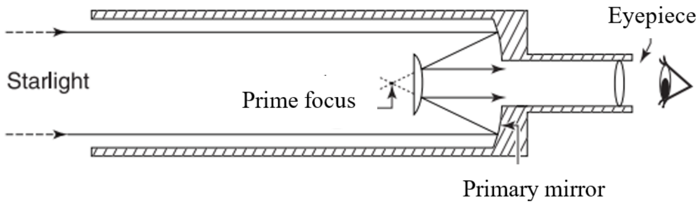


Figure 2.3: Cassegrain reflecting telescope.

2.4 F-number

Telescopes are often described by both their aperture size and **f-number**. The f-number is the ratio of the focal length of the main lens or mirror to the aperture. These specifications are important because the brightness, size, and clarity of the image produced by a telescope depend on the aperture and focal length of its main lens or mirror. For example, a "150-*mm* (6-*inch*), $f/8$ reflector" means the primary mirror is 150 *mm* (6-*inches*) in diameter and has a focal length of 1200 *mm* (8×150), or 48 inches (8×6).

2.5 Images

All stars except our Sun are so far away that they appear as dots of light in a telescope. The Moon and planets appear as small disks. **Image size** is proportional to the focal length of the telescope's main lens or mirror.

For example, a mirror with a focal length of 2.5 *m* (100 *inches*) produces an image of the Moon that measures about 2.5 *cm* (1 *inch*) across. You know that the 5 *m* (200-*inch*), $f/3.3$ mirror has a focal length of 16.5 *m* (660-*inches*), which is over six times as long. Hence, it produces an image of the Moon that is about six times as big or 15 *cm* (6-*inches*) across.

Lenses and mirrors form real images that are upside down. (A real image is formed by the actual convergence of light rays.) Since inverted images do not matter in astronomical work and righting them would require additional light-absorbing optics, nothing is done to turn images upright in telescopes.

2.6 Lens and the focal length

In a camera with a lens, the image will be in focus only at a fixed distance F from the lens. The distance F to the **focal plane** depends on the shape of the lens, as well as on its refractive index. For lenses, a useful parameter is the focal ratio $f = F/D$, where D is the diameter of the lens. The size of the image produced is not affected by the diameter D of the lens but only by the focal length F.

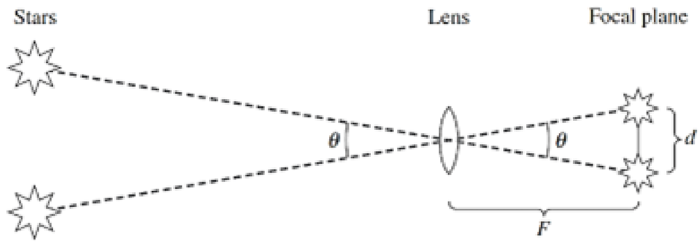


Figure 2.4: Focal plane.

In the $figure$ 2.4, two stars separated by a small angle θ on the sky have images that are separated by a physical distance d on the focal plane. Another useful parameter, in addition to the focal length, is the scale of the image on the focal plane, known for historical reasons as the plate scale. Specifically, an angular distance θ on the celestial sphere is related to a physical distance d on the image plane by the plate scale s:

$$\theta[arcsec] = s[arcsec/mm].d[mm], \tag{2.1}$$

we can also write this as:

$$\theta[radians] = \frac{d}{F}, \tag{2.2}$$

and therefore:

$$\theta[arcsec] = \theta[radians].\frac{180°}{\pi[radians]}.\frac{3600arcsec}{1°} = 206265(\frac{d}{f}), \tag{2.3}$$

we have a relationship between the plate scale s and the focal length F:

$$s[arcsec/mm] = \frac{206265}{F[mm]}. \tag{2.4}$$

The human eye, for instance, has a focal length $F \approx 17\ mm$, and hence a “plate scale” $s \approx 12,100\ arcsec/mm$, or $s \approx 3.4°/mm$; when you look at the full Moon, its image covers an area of your retina less than 0.15 mm across. Large astronomical telescopes have focal lengths that are more conveniently expressed in meters than in millimeters.

For these big telescopes, we may write:

$$s[arcsec/mm] = \frac{206.265}{F[m]} = \frac{206.265}{fD[m]}, \tag{2.5}$$

where f is the focal ratio, and D is the diameter of the telescope's aperture. As an example, the famous "forty-inch" Yerkes Telescope (at Williams Bay, Wisconsin) has an aperture $D = 1.02\ m$ and a focal ratio $f = 19$. The plate scale of the Yerkes Telescope is thus:

$$s = \frac{206.265}{19 \times (1.02)} arcsec/mm = 10.6\ arcsec/mm, \tag{2.6}$$

therefore an image of the full Moon produced by the Yerkes Telescope is 170 mm across, about the size of a salad plate.

The major optical component of a *refracting telescope* is the primary or objective lens of focal length f_{obj}. The purpose of the objective lens is to collect as much light as possible and with the greatest possible resolution, bringing the light to a focus at the focal plane. A photographic plate or other detector may be placed at the focal plane to record the image, or the image may be viewed with an eyepiece, which serves as a magnifying glass. The eyepiece would be placed at a distance from the focal plane equal to its focal length, f_{eye}, causing the light rays to be refocused at infinity. The figure below shows the path of rays coming from a point source lying off the optical axis at an angle θ. The rays ultimately emerge from the eyepiece at an angle ϕ from the optical axis. The angular magnification produced by this arrangement of lenses can be shown to be:

$$m = \frac{f_{obj}}{f_{eye}}. \tag{2.7}$$

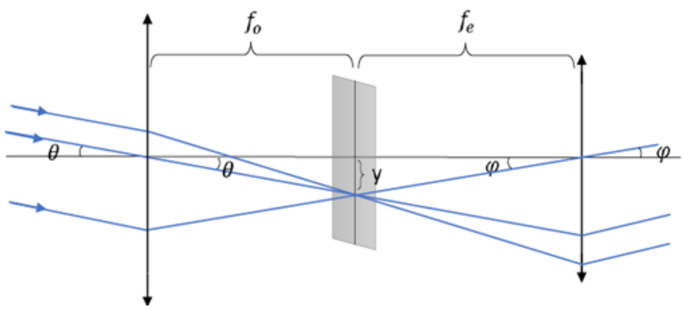


Figure 2.5: Telescope Magnification

In astronomy, the field of view is the amount of sky you can see, whether with your unaided vision, binoculars, or a telescope. If you had eyes on all sides of your head, you would have a 360° field of view. (Some insects actually do!) If you include peripheral vision, your naked eye field of view is nearly 180°, but with varying quality across this field. A telescope will have a much smaller field of view, but it has significant advantages, such as greater magnification and light-gathering power.

Field of view (FOV) is the diameter of a region of the sky that you can see using a specific instrument. The FOV would change with the magnification of the telescope you are using.

Figure 2.6: Field of view

2.7 Telescope resolving power

The **resolving power** of a telescope is defined as the ability of a telescope to distinguish objects with a small angle between them. This is known as the **theoretical resolving power** of the instrument. If the telescope is of good design and in correct adjustment, it should be possible to achieve this theoretical value. It should be possible to resolve two stars if they are separated by an angle (in radians) greater than:

$$\alpha = 1.22\frac{\lambda}{D}. \tag{2.8}$$

This value is known as the theoretical angular resolving power of the telescope. It can be seen that the resolving power is inversely proportional to the diameter of the objective. We take a value of 5500 *Angstroms* for λ being the effective wavelength for visual observations.

CHAPTER 3

Observing the Universe

sec:observing

Astronomy is ultimately an observational science. From the study of the life cycles of stars, to the dynamics of galaxies, to the detection of cosmic expansion, observations form the foundation of astrophysics. In this chapter we will focus on themes frequently emphasized in the International Olympiad on Astronomy and Astrophysics (IOAA): stellar evolution, galaxies, Doppler and redshift, Hubble's Law, and planetary motion.

3.1 Stellar Evolution

Stellar evolution refers to the life cycle of a star, from its birth within interstellar gas to its final stages as a white dwarf, neutron star, or black hole. This subject is central to both astrophysics and Olympiad-level problem solving: many theoretical, data analysis, and observational questions involve stellar properties, timescales, and end states.

Star Formation

Stars form in giant molecular clouds (GMCs), which are cold (10–50 K), dense (10^2–10^5 cm^{-3}), and primarily composed of molecular hydrogen.

- **Jeans instability:** collapse begins if the cloud's mass exceeds the Jeans mass
$$M_J \sim \frac{5kT}{2Gm_p}\left(\frac{3}{4\pi\rho}\right)^{1/2}. \tag{3.1}$$

- **Free-fall time:** the timescale for gravitational collapse is
$$t_{\text{ff}} \sim \sqrt{\frac{3\pi}{32G\rho}}. \tag{3.2}$$

During collapse, fragmentation produces protostars. Accretion disks and bipolar outflows are common features; angular momentum conservation plays a critical role.

Pre-Main Sequence and Hydrostatic Equilibrium

As a protostar contracts, gravitational potential energy converts into heat (Kelvin–Helmholtz contraction).

$$t_{\mathrm{KH}} \sim \frac{GM^2}{RL}. \tag{3.3}$$

For the Sun, $t_{\mathrm{KH}} \sim 3 \times 10^7$ years, consistent with the idea that nuclear fusion must power the Sun on longer timescales.

Hydrostatic equilibrium requires

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}, \tag{3.4}$$

with energy transport by radiation or convection depending on opacity.

Main Sequence Phase

The main sequence (MS) is characterized by hydrogen fusion in the core.

- **Mass–luminosity relation (IOAA favorite):**

$$L \propto M^{3.5} \quad (0.5M_\odot \lesssim M \lesssim 10M_\odot). \tag{3.5}$$

- **Stellar lifetime on the main sequence:**

$$t_{\mathrm{MS}} \sim 10^{10}\,\mathrm{yr}\left(\frac{M}{M_\odot}\right)^{-2.5}. \tag{3.6}$$

High-mass stars ($M > 8M_\odot$) live only a few Myr, while low-mass red dwarfs can survive for trillions of years.

Low-Mass Stars: Red Dwarfs and White Dwarfs

- **Red dwarfs** ($M < 0.5M_\odot$): fully convective, slowly burn hydrogen. No red dwarfs have yet reached the end of their lives in the universe's history.
- **White dwarfs:** degenerate remnants of stars with $M \lesssim 8M_\odot$. Supported by electron degeneracy pressure; radius decreases with mass.

The maximum mass is the **Chandrasekhar limit**:

$$M_{\mathrm{Ch}} \approx 1.44M_\odot. \tag{3.7}$$

Intermediate and High-Mass Stars

Stars with $M > 8M_\odot$:

- Evolve off the MS to red supergiants.
- Fuse heavier elements (He $\rightarrow$ C $\rightarrow$ O $\rightarrow$ ... up to Fe).

- End in **core-collapse supernovae** (Type II, Ib, Ic).

Observationally, IOAA has tested knowledge of:

- HR diagram tracks of stars of different masses.
- Cepheid variables as standard candles (period–luminosity relation).
- Energy released in a core collapse compared to binding energy of a neutron star.

Supernovae and Compact Remnants

- **Type Ia supernovae:** thermonuclear explosion of a white dwarf in a binary.
- **Type II/Ib/Ic:** collapse of massive cores.
- **Neutron stars:** supported by neutron degeneracy pressure; radius $\sim$ 10–15 km. Pulsars are rotating, magnetized neutron stars.
- **Black holes:** if core mass exceeds $\sim 3M_\odot$, collapse is inevitable.

The Schwarzschild radius is

$$R_s = \frac{2GM}{c^2}. \tag{3.8}$$

Key Olympiad Connections

- Estimating stellar lifetimes via mass–luminosity relation.
- Order-of-magnitude calculations of t_{KH} and t_{MS}.
- White dwarf radii from mass–radius relation.
- Jeans criterion in star formation.
- Energetics of supernovae and binding energies of compact objects.

3.2 Galaxies

Galaxies are vast gravitationally bound systems of stars, gas, dust, and dark matter. They come in a variety of morphologies and play a crucial role in cosmology and astrophysics. Understanding galaxies is also essential for Olympiad-level astronomy, as many problems involve galaxy dynamics, mass estimates, and cosmological applications.

Types of Galaxies

- **Elliptical galaxies:** Smooth, ellipsoidal systems, generally containing old, red, low-mass stars with little interstellar gas. They are often found in clusters and classified from E0 (nearly spherical) to E7 (highly elongated).
- **Spiral galaxies:** Disk-shaped galaxies with spiral arms, containing both young and old stars. They are gas-rich, active sites of star formation. The Milky Way and Andromeda are spiral galaxies. Barred spirals (SB) contain a central bar-like structure.
- **Irregular galaxies:** Lacking a defined structure, usually small and gas-rich. They are often satellites of larger galaxies and can be distorted by tidal forces. Examples: the Large and Small Magellanic Clouds.

Size and Mass Ranges

- Dwarf galaxies: masses $\sim 10^7$–$10^9 M_\odot$, containing 10^7–10^9 stars.
- Spirals: masses $\sim 10^{11} M_\odot$, sizes of tens of kiloparsecs.
- Giant ellipticals: masses up to $\sim 10^{13} M_\odot$, sizes ~ 100 kpc.

The largest known elliptical galaxy, IC 1101, has a diameter of over 6 million light-years and contains trillions of stars.

Galaxy Rotation and Dark Matter

Rotation curves of spiral galaxies provide one of the strongest pieces of evidence for dark matter.

- Observed: rotational velocity $v(r)$ remains approximately flat at large radii.
- Newtonian expectation: $v(r) \propto r^{-1/2}$ if mass is concentrated near the center.

This discrepancy suggests the presence of an extended dark matter halo. The dynamical mass inside radius r is estimated by

$$M(r) = \frac{v(r)^2 r}{G}. \tag{3.9}$$

Tully–Fisher Relation

For spiral galaxies, there is an empirical correlation between the galaxy's luminosity L and its maximum rotation velocity $v_{\max}$:

$$L \propto v_{\max}^{\alpha}, \tag{3.10}$$

with $\alpha \approx 4$. This relation is used as a distance indicator in extragalactic astronomy and has been tested in IOAA problems.

Mass-to-Light Ratio

The mass-to-light ratio (M/L) is a useful measure of how much dark matter a galaxy or cluster may contain:

$$\Upsilon = \frac{M}{L}. \tag{3.11}$$

For the Sun, $\Upsilon_\odot \equiv 1\, M_\odot/L_\odot$. Typical values:

- Spirals: $\Upsilon \sim 5$–10
- Ellipticals: $\Upsilon \sim 10$–20
- Clusters: $\Upsilon \sim 100$ or more, indicating large amounts of dark matter.

Supermassive Black Holes

Most galaxies, including the Milky Way, contain a supermassive black hole (SMBH) at their centers:

- Masses: 10^6–$10^{10} M_\odot$.
- Example: Sagittarius A* at the center of the Milky Way has a mass of $\sim 4 \times 10^6 M_\odot$.

Accretion onto SMBHs powers quasars and active galactic nuclei (AGN). Observations of SMBHs are often tested at IOAA (e.g., black hole shadows, Eddington luminosity).

Galaxy Evolution

Galaxies form hierarchically in the ΛCDM cosmological model: small protogalaxies merge to form larger systems. Key processes:

- **Mergers:** drive morphological transformations (e.g., spirals merging into ellipticals).
- **Starbursts:** episodes of intense star formation triggered by interactions.
- **Chemical enrichment:** successive generations of stars enrich the interstellar medium with heavier elements ("metals").

Observations show that galaxies in the early universe were more irregular, gas-rich, and star-forming compared to present-day galaxies.

Key Olympiad Connections

- Estimating galaxy mass from rotation curves and $M(r) = v^2 r/G$.
- Using Tully–Fisher relation to compute extragalactic distances.
- Order-of-magnitude estimates of dark matter fractions in galaxies.
- Supermassive black hole mass estimates from stellar orbital velocities.

3.3 Doppler Effect and Redshift

The Doppler effect describes the change in wavelength or frequency of a wave due to relative motion between source and observer. For astronomy, it is crucial in determining radial velocities of stars, galaxies, and exoplanets.

Classical Doppler Effect

For a source moving with radial velocity v_r much smaller than the wave speed c, the fractional shift in wavelength is

$$\frac{\Delta\lambda}{\lambda_{\text{rest}}} = \frac{\lambda_{\text{obs}} - \lambda_{\text{rest}}}{\lambda_{\text{rest}}} = \frac{v_r}{c}. \tag{3.12}$$

- If $v_r > 0$ (object receding), $\lambda_{\text{obs}} > \lambda_{\text{rest}}$ (redshift).
- If $v_r < 0$ (object approaching), $\lambda_{\text{obs}} < \lambda_{\text{rest}}$ (blueshift).

Relativistic Doppler Effect

For velocities comparable to c, time dilation must be taken into account. The exact formula is

$$z \equiv \frac{\Delta\lambda}{\lambda_{\text{rest}}} = \sqrt{\frac{1 + v_r/c}{1 - v_r/c}} - 1. \tag{3.13}$$

This reduces to the classical form $z \approx v_r/c$ for $v_r \ll c$.

Redshift Parameter

In astronomy, the dimensionless redshift z is defined as

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{rest}}}{\lambda_{\text{rest}}}. \tag{3.14}$$

For small z, $v \approx cz$, but at high z (cosmological scales) redshift arises from the expansion of space rather than Doppler motion alone.

Applications in Astronomy

- **Binary stars:** Periodic Doppler shifts in spectral lines measure orbital velocities and mass ratios.
- **Exoplanets:** Radial velocity method uses tiny Doppler shifts ($\sim$ m/s precision).
- **Galaxies:** Measuring redshift provides recessional velocity and, via Hubble's Law, distance.
- **Quasars and cosmology:** Very high redshifts ($z > 6$) indicate light emitted when the universe was less than a billion years old.

—

3.4 Hubble's Law

In the 1920s, Edwin Hubble discovered that galaxies show a redshift roughly proportional to their distance. This implies that the universe as a whole is expanding.

$$v = H_0 D, \tag{3.15}$$

where

- v is the recession velocity (km/s),
- D is the galaxy's distance (Mpc),
- H_0 is the Hubble constant (~ 70 km/s/Mpc).

Estimating the Age of the Universe

Assuming constant expansion rate,

$$t_{\text{Hubble}} \approx \frac{1}{H_0}. \tag{3.16}$$

For $H_0 \approx 70$ km/s/Mpc,

$$t_{\text{Hubble}} \sim 14\,\text{Gyr},$$

consistent with modern estimates from cosmology.

IOAA-style Connections

- Deriving galaxy distances from measured redshift z and Hubble's Law.
- Estimating the Hubble time t_H from a given H_0.
- Problems involving recessional velocity and lookback time.
- Recognizing when to use relativistic vs. classical Doppler formula.

3.5 Planetary Motion

The apparent motion of planets in the sky is the result of both their orbital motion around the Sun and the Earth's own motion. Understanding planetary configurations is a classic topic for Olympiad-level astronomy, especially for observational and theoretical questions.

Direct and Retrograde Motion

Most of the time, planets appear to move eastward (direct motion) relative to the background stars. Occasionally, outer planets (e.g., Mars, Jupiter, Saturn) exhibit westward loops (retrograde motion) as the faster-moving Earth overtakes them in its orbit.

Inferior and Superior Planets

Planets are classified based on their orbital size relative to Earth:

- **Inferior planets:** Mercury and Venus, with orbits inside Earth's orbit. They never appear far from the Sun in the sky. Their maximum angular separation is called the **greatest elongation**.
  - Mercury: max elongation $\sim 28°$.
  - Venus: max elongation $\sim 47°$.
- **Superior planets:** Mars, Jupiter, Saturn, Uranus, Neptune. Their orbits lie outside Earth's orbit, so they can appear anywhere along the ecliptic relative to the Sun.

Planetary Configurations

Key configurations for Olympiad problems include:

- **Conjunction:** Planet aligned with the Sun.
- **Opposition:** For superior planets, occurs when Earth is between the planet and the Sun. Planet rises at sunset, visible all night.
- **Quadrature:** Superior planet appears 90° east or west of the Sun in the sky.
- **Inferior conjunction:** Inferior planet lies between Earth and Sun.
- **Superior conjunction (inferior planets):** The Sun lies between Earth and the planet.

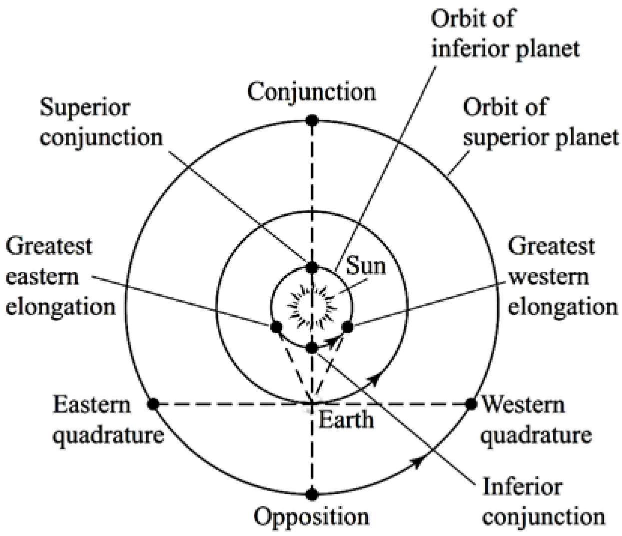


Figure 3.1: Planetary configurations: elongations, conjunctions, opposition, and quadrature.

Synodic and Sidereal Periods

The orbital motion of planets leads to two distinct periods:

- **Sidereal period (T):** Orbital period of a planet relative to the stars.
- **Synodic period (S):** Time between successive similar configurations (e.g., oppositions, conjunctions) as seen from Earth.

The relation between them is

$$\frac{1}{S} = \left| \frac{1}{T_{\text{planet}}} - \frac{1}{T_{\oplus}} \right|, \tag{3.17}$$

where $T_{\oplus} = 1$ year is Earth's orbital period.

IOAA-style Examples

- Given orbital periods, compute synodic periods for different planets.
- Use elongation angle and Earth–Sun distance to estimate orbital radius of an inferior planet.
- Geometry-based problems on retrograde loops of Mars or Jupiter.

CHAPTER 4

Spherical Astronomy

We have seen that the observer who views the heavens at night gets the impression that they are at the centre of a great hemisphere onto which the heavenly bodies are projected. The moon, planets, and stars seem to lie on this celestial hemisphere, their directions defined by the positions they have on its surface. For many astronomical purposes, the distances are irrelevant so the radius of the sphere can be chosen at will. The description of the positions of bodies on it, considering positional changes with time, necessarily involves the use of special coordinate and timekeeping systems. The relationship between the positions of bodies requires a knowledge of the geometry of the sphere. This branch of astronomy, called **spherical astronomy**, is in one sense the oldest branch of the subject, its foundations dating back at least 4000 years. Its subject matter is still essential and never more so than today when the problem arises of observing or calculating the position of an artificial satellite or interplanetary probe. We, therefore, begin by considering the geometry of the sphere.

4.1 Spherical geometry

The geometry of the sphere is made up of great circles, small circles, and arcs of these figures. Distances along great circles are often measured as angles since, for convenience, the radius of the sphere is made unity. A great circle is defined to be the intersection with the sphere of a plane containing the centre of the sphere. Since the centre is equidistant from all points on the sphere, the figure of intersection must be a circle. If the plane does not contain the centre of the sphere, its intersection with the sphere is a small circle.

We can draw infinite circles on a sphere, some may have a radius of the sphere (great circles) and others will have a smaller radius (small circles). In the figure on the right ANBM, CNDM, and APBQ are all great circles, while EFG is a small circle.

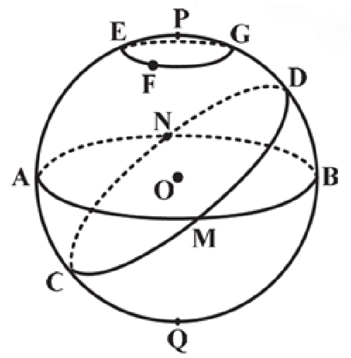


Figure 4.1: Great and small circles on a sphere

The area of the spherical triangle can be found by the equation:

$$S_{ABC} = (A + B + C - \pi)R^2, \tag{4.1}$$

where all angles must be written in terms of *Radians* and R is the radius of the sphere. Just as the formulas of plane trigonometry can be used to perform calculations in plane geometry, special trigonometrical formulas for use in spherical geometry can be established. There are many such formulas but four are more often used than any of the others. They are the relations between the sides and angles of a spherical triangle and are invaluable in solving the problems that arise in spherical astronomy.

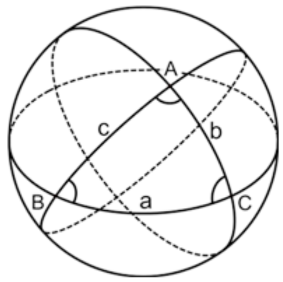


Figure 4.2: Area of a spherical triangle with angles of A, B, and C

ABC is a spherical triangle with sides AB, BC, and CA of lengths c, a, and b, respectively, and with angles $\angle CAB$,$\angle ABC$, and $\angle BCA$ hereafter referred to as angles A, B and C respectively. The four formulas are:

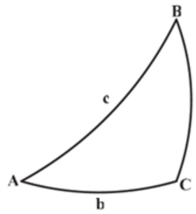


Figure 4.3: Sample spherical triangle

Sine formula:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \tag{4.2}$$

Cosine formula:

$$\begin{aligned}\cos a &= \cos b \cos c + \sin b \sin c \cos A \\ \cos b &= \cos a \cos c + \sin a \sin c \cos B \\ \cos c &= \cos a \cos b + \sin a \sin b \cos C\end{aligned} \tag{4.3}$$

Polar formula formula:

$$\begin{aligned}-\cos A &= \cos B \cos C + \sin B \sin C \cos a \\ -\cos B &= \cos A \cos C + \sin A \sin C \cos b \\ -\cos C &= \cos A \cos B + \sin A \sin B \cos c\end{aligned} \tag{4.4}$$

Four-parts formula:

$$\cos a \cos C = \sin a \cot b - \sin C \cot B \tag{4.5}$$

4.2 Position on the Earth's surface

To illustrate these concepts, we consider the Earth. Geographers have already shown us how to set up a coordinate system on a sphere; the system of **latitude** and **longitude** provides a coordinate system on the surface of the (approximately) spherical Earth. On the Earth, the north and south poles represent the points where the Earth's rotation axis passes through the Earth's surface. The **equator** is the great circle midway between the north and south pole, dividing the Earth's surface into a northern hemisphere and a southern hemisphere.

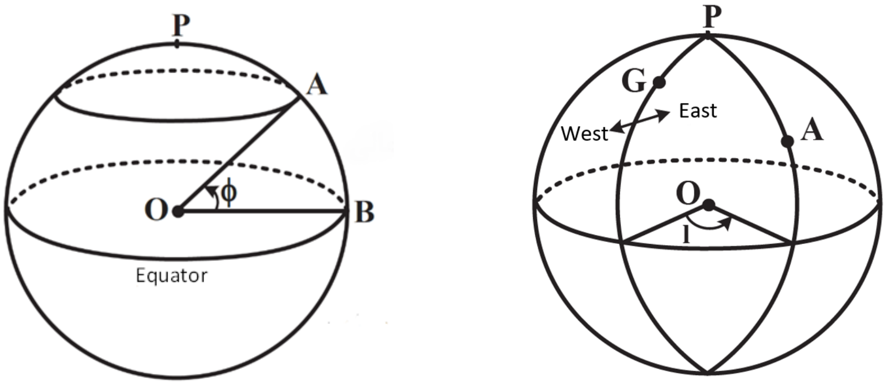


Figure 4.5: Longitude l

The latitude of a point on the Earth's surface is its angular distance from the equator, measured along a great circle perpendicular to the Earth's equator. Latitude is measured in degrees, arc-minutes, and arc-seconds, as is longitude. Thus, the use of latitude and longitude does not require knowing the size of the Earth in kilometers or any other unit of length. The longitude may be expressed in angular measure or in time units related to each other by the table on the right.

$360^\circ = 24^h$	
$1^\circ = 4^m$	$1^h = 15^\circ$
$1' = 4^s$	$1^m = 15'$
$1'' = (1/15)^s$	$1^s = 15''$

Figure 4.6: Unit conversions used in spherical astronomy

4.3 The horizontal (alt-azimuth) system

It is convenient to imagine a sphere at a great distance ("infinity") upon which all stars lie. This is called the **celestial sphere**. The positions of stars on this sphere may be specified with two angles, analogous to the way latitude and longitude specify a position on the earth's surface. This celestial sphere is an artificial construction; stars are not all at the same distance. Stars in our own Galaxy range in the distance from 4 light-years to more than 50000 light-years from the Earth. Nevertheless, the concept of the celestial sphere is useful for charting the sky as one sees it.

One such coordinate system on the celestial sphere is based on an observer's horizon and hence is called the horizon coordinate system. In this system, the latitude-like coordinate is the altitude, defined as the angle of a celestial object above the horizon circle. The zenith (the point directly overhead) is at an altitude of 90°. Points on the horizon circle are at an altitude of 0°. The nadir is at an altitude of −90°, but in practice, negative altitudes are seldom used, since they represent objects that are hidden by the Earth. The longitude-like coordinate in the horizon coordinate system is called the azimuth

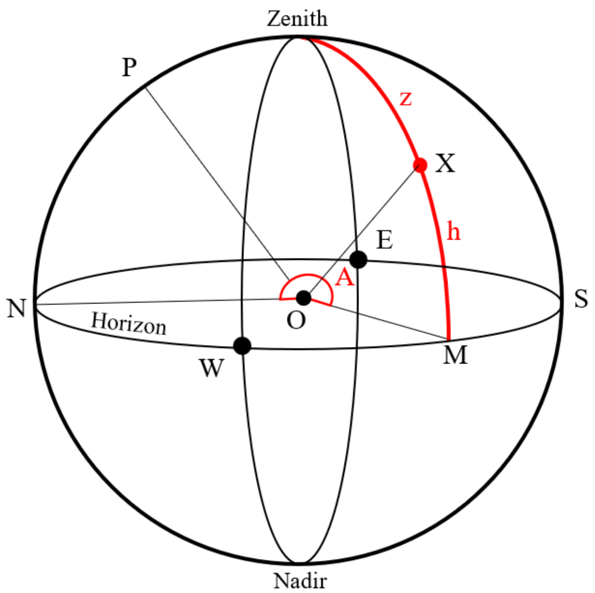


Figure 4.7: Horizontal(alt-azimuth) coordinate system

In the above figure, X is the position of the star. The arc of $\widehat{XM} = h$, where h is the altitude. Therefore, the zenith distance is $\widehat{XZ} = 90° - h = z$. The azimuth of this star is the red angle shown in the figure: $A = 360° - \widehat{NOM}$. Azimuth is usually expressed from North to East. But if the star is located in the western hemisphere (like the star in the figure), we can express the azimuth from North to West: $A = \widehat{NOM}\, W$.

For any point on the celestial sphere, half a great circle can be drawn from the zenith, through the point in question, to the nadir. The half-circle that runs through the north point on the horizon circle acts as the ı*primemeridian*ȷ in the horizon coordinate system. The azimuth is measured in degrees running from north to east. An object due north of an observer has an azimuth of 0°, an object due east has an azimuth of 90°, and so forth. If you know the altitude and azimuth of any object in your horizon coordinate system, you know where to point your telescope to see it.

If we consider the figure below for an observer in a particular latitude of ϕ, the direction of rotation of the Earth is P_1, and since the north celestial pole (NCP) is in distant, P_2 will be the direction for that the person will see the north celestial pole. It is depicted that the altitude of the pole is equal to the latitude of the observer.

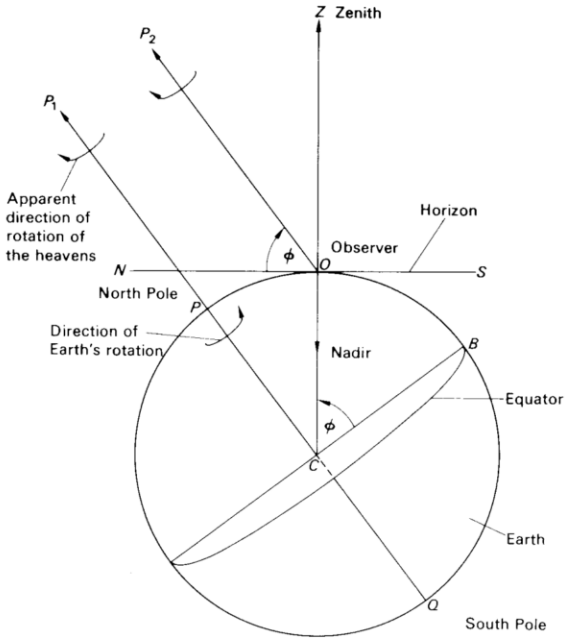


Figure 4.8: altitude of NCP for the observer on Earth

One shortcoming of the horizon coordinate system is that every observer on Earth has a different, unique horizon and hence has a different, unique horizon coordinate system. A star that is near the zenith (*altitude* $\approx 90°$) for an observer in Buenos Aires will be near the nadir (altitude -90°) for an observer in the antipodal city of Shanghai. To describe the positions of objects on the celestial sphere, it is useful to have a coordinate system that all astronomers, regardless of location, can agree on, just as geographers all agree to use latitude and longitude to describe positions on the Earth.

4.4 The equatorial system

To build a coordinate system that works for everyone on Earth, we start by projecting the Earth's poles and equator outward onto the celestial sphere. The Earth's rotation axis, which passes through the north and south poles of the Earth, intersects the celestial sphere at the **north celestial pole** (labeled as NCP) and the **south celestial pole** (labeled as SCP). The north celestial pole is at the zenith for an observer at the Earth's north pole; more generally, for an observer at a latitude north of the equator, it will be at an altitude of ϕ and azimuth of 0°. The projection of the Earth's equator onto the celestial sphere is called the **celestial equator**. The celestial equator passes through the zenith for an observer on the Earth's equator.

On the Earth's surface, a point's latitude is its angular distance north or south of the equator. Similarly, on the celestial sphere, a point's **declination** (δ) is its angular distance north or south of the celestial equator. For points north of the celestial equator, the declination is positive ($0° < \delta \leq 90°$), and for points south of the celestial equator, the declination is negative ($-90° \leq \delta < 0°$).

Right ascension α is analogous to longitude and is measured eastward along the celestial equator from the vernal equinox (γ) to its intersection with the object's hour circle (the great circle passing through the object being considered and through the north celestial pole). Right ascension is traditionally measured in hours, minutes, and seconds. The coordinates of the right ascension and declination are also indicated in the figure below. Since the equatorial coordinate system is based on the celestial equator and the vernal equinox, changes in the latitude and longitude of the observer do not affect the values of right ascension and declination. Values of and are similarly unaffected by the annual motion of Earth around the Sun.

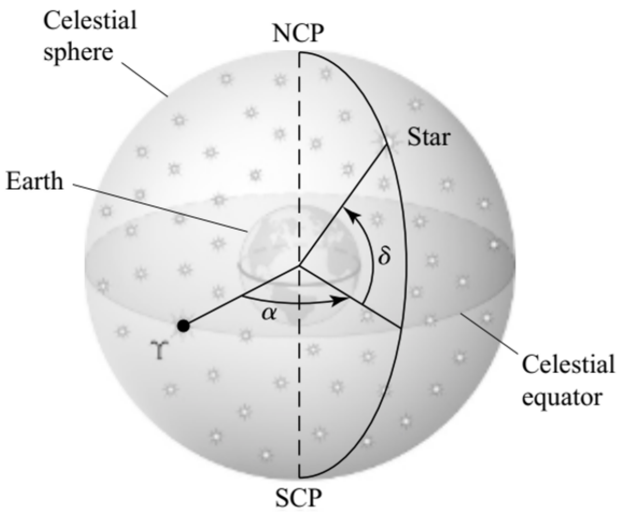


Figure 4.9: Equatorial coordinate system

We used a unique point to define the Azimuth angle in horizontal coordinates, that specific point is North. Having that in mind, we also know that the NCP is pointing towards the North. We can draw both coordinates on a sphere for an observer.

If we sketch both coordinates on a single sphere, then the celestial equator intersects the horizon circle in two points West and East. Points P and Z are the poles of the celestial equator and the horizon respectively. But W lies on both these great circles so that W is 90° from the points P and Z. Hence, W is a pole on the great circle ZPN and must, therefore, be 90° from all points on it—in particular from N and S. Hence, it is the west point. By a similar argument, E is the east point. Any great semicircle through P and Q is called a *meridian.* The meridian through the celestial object X is the great semicircle $PXBQ$ cutting the celestial equator in B.

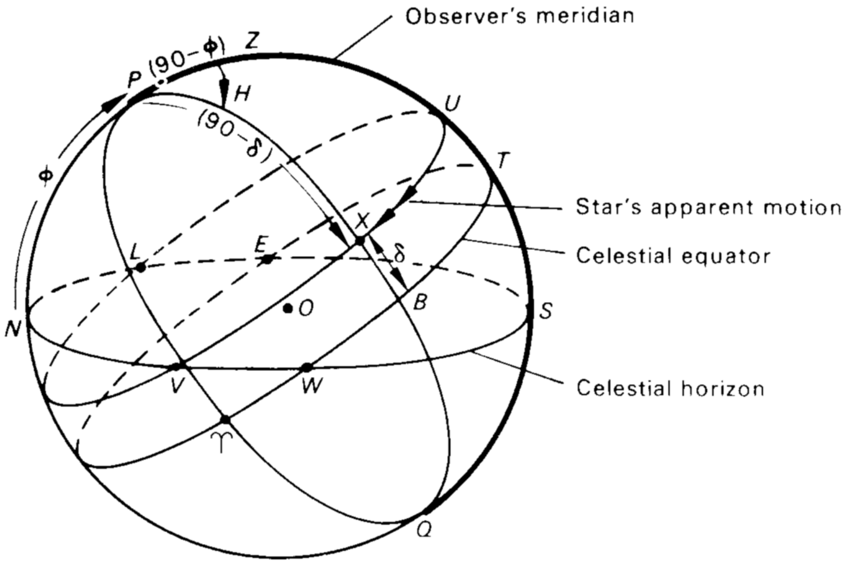


Figure 4.10: Equatorial coordinate system for an observer on Earth

In particular, the meridian $PZTSQ$ indicated because of its importance by a heavier line is the **observer's meridian**. An observer viewing the sky will note that all natural objects rise in the east, climbing in altitude until they **transit** across the observer's meridian then decrease in altitude until they set in the west. A star, in fact, will follow a small circle parallel to the celestial equator in the arrow's direction. Such a circle (UXV in the diagram) is called a **parallel of declination** and provides us with one of the two coordinates in the equatorial system. The **declination**, δ, of the star is the angular distance in degrees of the star from the equator along the meridian through the star. It is measured north and south of the equator from 0° to 90°, being taken to be positive when north. The declination of the celestial object is thus analogous to the latitude of a place on the Earth's surface, and indeed the latitude of any point on the surface of the Earth when a star is in its zenith is equal to the star's declination.

A quantity called the **north polar distance** of the object (X in the figure) is often used. It is the arc PX.

Obviously,

$$\textit{north polar distance} = 90^\circ - \textit{declination}.$$

It is to be noted that the north polar distance can exceed 90°. The star, then, transits at U, sets at V, rises at L and transits again after one rotation of the Earth. The second coordinate recognizes this. The angle ZPX is called the **hour angle**, t, of the star and is measured from the observer's meridian westwards (for both north and south hemisphere observers) to the meridian through the star from 0^h to 24^h or from 0° to 360°. Consequently, the hour angle increases by 24^h each sidereal day for a star. Having both coordinates on the sphere, using Zenith, the North celestial pole, and the star (three points) we are able to create a spherical triangle (figure below). We need to use spherical trigonometry to solve any spherical triangle.

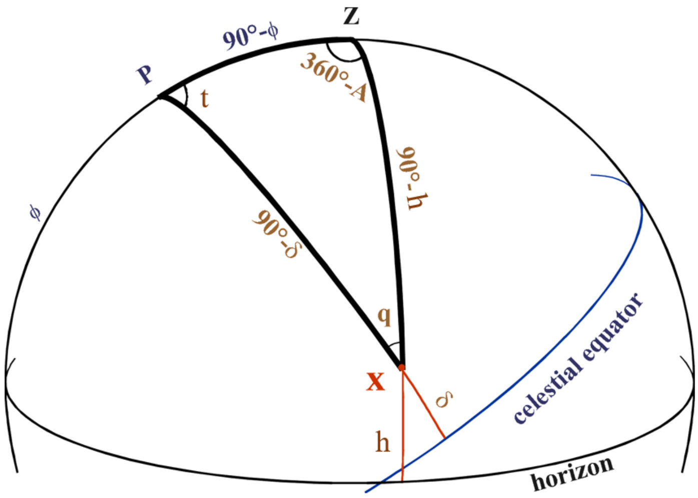


Figure 4.11: Coordinate system conversion

A common problem in spherical astronomy is obtaining a star's coordinates in one system, given the coordinates in another system. The observer's latitude is usually known. For example, we may want to calculate the hour angle of t and declination δ of a body when its azimuth (east of north) and altitude are A and h. Assume the observer has a latitude ϕ. We start by writing the cosine formula:

$$\begin{aligned} \cos PX &= \cos PZ \cos ZX + \sin PZ \sin ZX \cos PZX \\ \longrightarrow \quad \sin\delta &= \sin\phi \sin h + \cos\phi \cos h \cos A \end{aligned} \tag{4.6}$$

By using the cosine formula again:

$$\begin{aligned} \cos ZX &= \cos PZ \cos PX + \sin PZ \sin PX \cos ZPX \\ \longrightarrow \quad \sin a &= \sin\phi \sin\delta + \cos\phi \cos\delta \cos t \end{aligned} \tag{4.7}$$

You could also use four-parts or sine law to solve the spherical triangle. Based on the known parameters in the triangle, you should decide which formulas to use in order to solve the triangle.

4.5 The ecliptic system

The orbital plane of the Earth, the *ecliptic*, is the reference plane of another important coordinate frame. The ecliptic can also be defined as the great circle on the celestial sphere described by the Sun over the course of one year. This frame is used mainly for planets and other bodies of the solar system. The orientation of the Earth's equatorial plane remains invariant, unaffected by its annual motion. In spring, the Sun appears to move from the southern hemisphere to the northern one. The time of this remarkable event as well as the direction of the Sun at that moment is called the vernal equinox. At the *vernal equinox*, the Sun's right ascension and declination are zero.

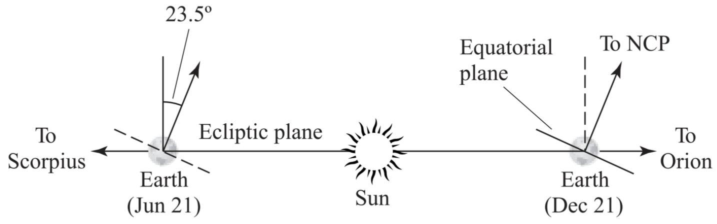


Figure 4.12: The plane of Earth's orbit seen edge-on

The two quantities specifying the position of an object on the celestial sphere in this system are ecliptic longitude and ecliptic latitude. In *figure 4.13* below a great circle arc through the pole of the ecliptic K and the celestial object X meets the ecliptic in point D. Then the **ecliptic longitude**, λ, is the angle between γ and D, measured from 0° to 360° along the ecliptic in the eastwards direction, that is in the direction in which right ascension increases. The **ecliptic latitude**, β, is measured from D to X along the great circle arc DX, being measured from 0° to 90° north or south of the ecliptic. It should be noted that the north pole of the ecliptic, K, lies in the hemisphere containing the north celestial pole. It should also be noted that ecliptic latitude and longitude are often referred to as **celestial latitude** and **longitude**.

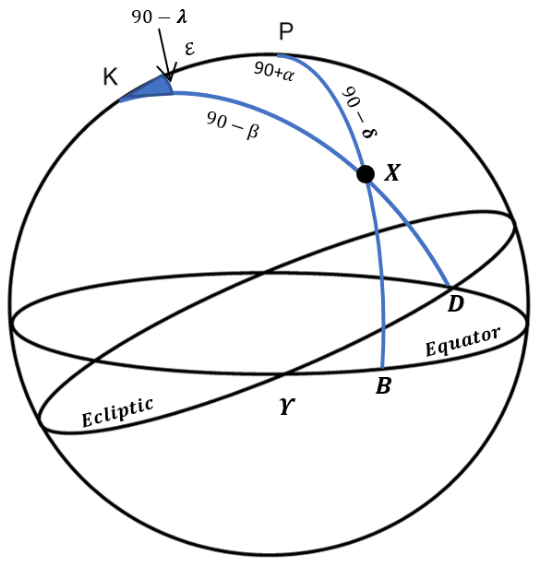


Figure 4.13: The Celestial sphere used for coordinate system conversion

Let's assume the equatorial coordinates of a star are known, and we want to determine its ecliptic coordinates. This means α and δ are given. Using the spherical triangle above, we can use cosine law:

$$\begin{aligned} &\cos(90-\beta) = \cos\epsilon\cos(90-\delta) + \sin\epsilon\sin(90-\delta)\cos(90+\alpha) \\ \longrightarrow\quad &\sin\beta = \cos\epsilon\sin\delta - \sin\epsilon\cos\delta\cos\alpha \end{aligned} \tag{4.8}$$

By using the cosine formula again:

$$\begin{aligned} &\cos(90-\delta) = \cos\epsilon\cos(90-\beta) + \sin\epsilon\sin(90-\beta)\cos(90-\lambda) \\ \longrightarrow\quad &\sin\delta = \cos\epsilon\sin\beta - \sin\epsilon\cos\beta\cos\lambda \\ \longrightarrow\quad &\sin\lambda = \tfrac{\sin\delta-\cos\epsilon\sin\beta}{\sin\epsilon\cos\beta} \end{aligned} \tag{4.9}$$

CHAPTER 5

Celestial Mechanics

By applying Newton's laws of motion and the law of universal gravitation, we are able to comprehend and analyze the complex movements of celestial objects within the solar system. The celestial objects that we can observe include the planets, comets, natural satellites, and man-made satellites that are orbiting around their respective planets. The analytical process is simplified by making two assumptions. The first assumption is that we only consider the gravitational force between the orbiting body, such as the Earth, and the central body, which is the Sun. We disregard the gravitational forces exerted by other celestial bodies, such as other planets, to focus solely on the interaction between the orbiting and central body. Secondly, we assume that the central body is significantly more massive than the orbiting body, enabling us to disregard the central body's motion caused by their mutual attraction. Although both objects actually orbit around their common center of mass, if one of the celestial bodies is much more massive than the other, the center of mass can be approximated to be at the center of the heavier object.

5.1 Newton's Law of Gravitation

Gravitational force is one of the four fundamental forces of nature, along with electromagnetic force, weak nuclear force, and strong nuclear force. It is the force that causes objects with mass to be attracted to one another. In this article, we will explore the concept of gravitational force and its mathematical description.

The English physicist Sir Isaac Newton was the first to describe the nature of gravitational force. He formulated his law of gravitation in 1687, which states that the force of attraction between two objects with masses m_1 and m_2, separated by a distance r, is given by:

$$F_G = G\frac{m_1 m_2}{r^2}, \tag{5.1}$$

where F_G is the gravitational force, G is the gravitational constant, and r is the distance between the two objects. The value of G is approximately 6.674×10^{-11} N $\mathrm{m}^2/\mathrm{kg}^2$.

Gravitational Field

The gravitational force can also be described in terms of a gravitational field. A gravitational field is a region of space where an object with mass experiences a force due to the presence of another object with mass. The gravitational field strength at a point in space is defined as the force per unit mass experienced by a small test mass placed at that point.
The gravitational field strength at a distance r from a point mass M is given by:

$$g = \frac{GM}{r^2}, \tag{5.2}$$

where G is the gravitational constant. The gravitational field strength is a vector quantity, pointing towards the point mass M.

Gravitational Potential Energy

Gravitational force is a conservative force, meaning that the work done by the force in moving an object from one point to another is independent of the path taken. The gravitational potential energy of an object at a point in space is the amount of work required to move the object from an infinite distance to that point, against the gravitational force.
The gravitational potential energy U of an object of mass m at a distance r from a point mass M is given by:

$$U = -\frac{GMm}{r}, \tag{5.3}$$

The negative sign indicates that the gravitational force is attractive, and the potential energy is lower at closer distances.

5.2 Linear Momentum

Linear momentum, also known as linear motion, is the product of an object's mass and its velocity. It is a vector quantity, meaning it has both magnitude and direction. The formula for linear momentum is:

$$p = mv, \tag{5.4}$$

where p is the linear momentum, m is the mass of the object, and v is its velocity.
Linear momentum is conserved in an isolated system, meaning that the total linear momentum of a system remains constant if no external forces act upon it. This principle is known as the law of conservation of linear momentum.

5.3 Angular Speed

Angular speed is the rate at which an object rotates or revolves about a fixed axis. It is a scalar quantity, meaning it has magnitude but no direction. The formula for angular speed is:

$$\omega = \frac{\theta}{t}, \tag{5.5}$$

where ω is the angular speed, θ is the angular displacement of the object, and t is the time taken for the object to complete the rotation.
Angular speed is measured in radians per second (rad/s). It is important to note that angular speed is not the same as linear speed, which is the distance traveled per unit time.

5.4 Angular Momentum

Angular momentum is the rotational equivalent of linear momentum. It is the product of an object's moment of inertia and its angular velocity. The formula for angular momentum is:

$$L = I\omega, \tag{5.6}$$

where L is the angular momentum, I is the moment of inertia of the object, and ω is its angular velocity.
Angular momentum can also be expressed as the product of the mass of the object, its tangential velocity, and the distance from the axis of rotation:

$$L = mrv, \tag{5.7}$$

where m is the mass of the object, v is its tangential velocity, and r is the distance from the axis of rotation.
Angular momentum is also a vector quantity, meaning it has both magnitude and direction. Its direction is perpendicular to the plane of rotation. The moment of inertia is a measure of an object's resistance to rotational motion and depends on both the mass and the distribution of mass relative to the axis of rotation.

Like linear momentum, angular momentum is conserved in an isolated system. This principle is known as the law of conservation of angular momentum. The law states that if no external torques act upon an isolated system, the total angular momentum of the system remains constant. Mathematically, this can be expressed as:

$$\frac{dL}{dt} = \tau_{net}, \tag{5.8}$$

where dL/dt is the rate of change of angular momentum and τ_{net} is the net external torque acting on the system. If there is no net external torque, then dL/dt is zero and the angular momentum of the system is conserved.

5.5 Conservation of Angular Momentum

The law of conservation of angular momentum has many important applications in physics. For example, it can be used to explain the behavior of spinning tops, the motion of planets around the sun, and the behavior of particles in quantum mechanics.

One important application of conservation of angular momentum is in understanding the behavior of rotating systems. For example, when an ice skater pulls their arms in, their moment of inertia decreases, causing their angular velocity to increase, and their angular momentum to remain constant. This principle is also used in designing objects such as satellites and gyroscopes, which rely on the conservation of angular momentum to maintain their stability and orientation in space.

Another important application of conservation of angular momentum is in the study of collisions. When two objects collide, their angular momentum may change due to external torques, such as friction. However, if the collision is elastic and there are no external torques, the total angular momentum of the system will remain constant.

Overall, the law of conservation of angular momentum is a fundamental principle in physics that helps to explain the behavior of rotating systems and the interactions between objects in motion.

5.6 Kepler's laws

Kepler's laws, which describe the motions of planets, were originally deduced by Johannes Kepler from observations of the planet Mars. However, these laws can also be derived from Isaac Newton's laws of motion and his law of gravitation, which provide an empirical basis for understanding planetary motion. Additionally, the application of Newton's laws of motion and law of universal gravitation extends beyond just the study of celestial bodies in our solar system. These laws can be applied to the study of the universe as a whole, including the behavior of galaxies and the evolution of the universe itself. They also have practical applications, such as in the design and operation of spacecraft and satellites. By understanding how gravity works and how objects move in space, scientists and engineers can plan and execute space missions with incredible precision, including everything from sending probes to explore distant worlds to placing satellites in orbit for communication and navigation purposes.

1. Kepler's First Law: All planets follow elliptical orbits with the Sun at one of the two foci. Newton realized that there is a direct mathematical relationship between inverse-square ($\frac{1}{r^2}$) forces and elliptical orbits. Figure 4.1 illustrates a typical elliptical orbit, where the orbiting body is located at polar coordinates (r, θ) and the origin is at the central body. An elliptical orbit is characterized by two parameters: the semimajor axis a and the eccentricity e. The distance from the center of the ellipse to either focus is ea. A circular orbit is a special case of an elliptical orbit with $e = 0$, where the two foci merge to a single point at the center of the circle. For example, Earth follows an elliptical orbit with an eccentricity of approximately 0.0167.

The maximum distance R_a of the orbiting body from the central body is indicated by the prefix $apo-$ (or sometimes $ap-$), as in aphelion (the maximum distance from the Sun) or apogee (the maximum distance from Earth). Similarly, the closest distance R_p is indicated by the prefix $peri-$, as in perihelion or perigee. As you can see from Figure 4.1:

$$R_a = a(1 + e), R_p = a(1 - e). \tag{5.9}$$

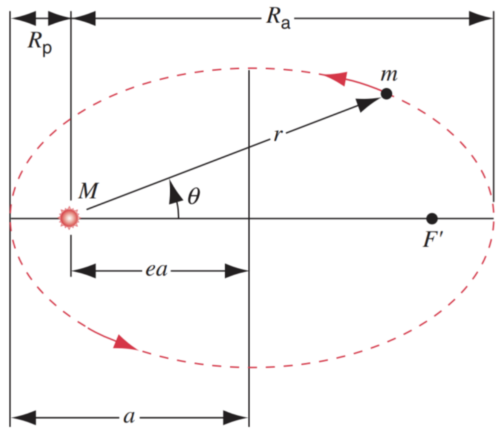


Figure 5.1: A planet of mass m moving in an elliptical orbit around the Sun with mass M

And for circular orbits $R_a = R_p$.

2. The Law of Areas: dictates that, during equal intervals of time, the imaginary line that connects a planet to its central star will cover equal areas. Figure 4.2 serves to visually demonstrate this concept, and implies that an orbiting object will move with greater velocity when it is nearer to the central body than when it is further away. It can be proven that the Law of Areas is in fact equivalent to the Law of Conservation of Angular Momentum.

If we examine the small area increment A that is traversed during a time interval t, as illustrated in Figure 4.2, we can see that the area of the triangular wedge is roughly equal to half of its base, $r\Delta\theta$ multiplied by its height r. We can then calculate the rate at which this area is swept out:

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{2} r^2 \frac{\Delta\theta}{\Delta t} = \frac{1}{2} r^2 \omega \tag{5.10}$$

If we make the assumption that the more massive body M can be considered stationary, then the angular momentum of the orbiting body m can be described relative to the origin at the central body as:

$$L_z = I\omega = mr^2\omega \tag{5.11}$$

Thus:

$$\frac{dA}{dt} = \frac{L_z}{2m} \tag{5.12}$$

If the M and m system is isolated and there is no external torque acting on it, then the angular momentum L_z remains constant. This means that the derivative of the area A with respect to time t is also constant, as stated in the equation. Consequently, during each interval of time dt, the line connecting m and M sweeps out an equal area dA, which confirms Kepler's second law. The increase in speed of a comet as it passes close to the Sun is an example of this effect and is directly related to the law of conservation of angular momentum.

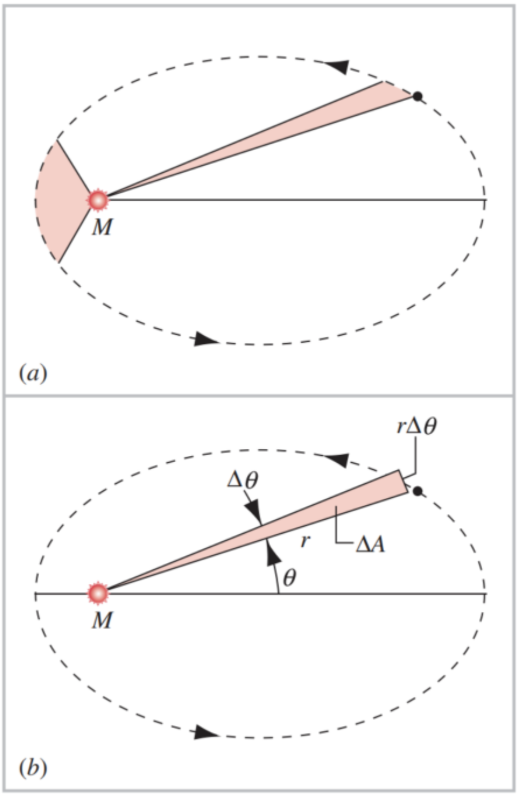


Figure 5.2: (a) The law of areas is demonstrated by the equal shaded areas, which are traversed by a line connecting a planet to the Sun in equal time intervals. (b) During a time interval t, the line connecting a planet to the Sun sweeps through an angle theta (θ) while covering an area A.

2. The Law Periods: One of the fundamental laws of planetary motion is that the square of a planet's orbital period around the Sun is directly proportional to the cube of its mean distance from the Sun. This relationship holds true for circular orbits as well. It is important to note that the force of gravity acts as the centripetal force for the circular motion. Therefore, the planet's acceleration is always directed towards the center of the orbit. This allows us to use the principles of circular motion to derive this relationshi:

$$\frac{GMm}{r^2} = m\frac{v^2}{r}. \tag{5.13}$$

Then replacing the speed v with $4\pi r/T$, where T is the rotational period (the time for a full orbit), we obtain:

$$T^2 = (\frac{4\pi^2}{GM})r^3. \tag{5.14}$$

The same outcome can be achieved for orbits that are elliptical, where the radius r is substituted with the semi-major axis a. The constant ratio between

T^2 and a^3 is determined by the quantity $4\pi^2/GM$, which applies to all planets orbiting the Sun. This relationship is confirmed by the data presented in Table 4.1. By measuring T and a for an orbiting body, we can calculate the mass of the central body, regardless of the orbiting body's mass. It should be noted that this method does not provide any information about the mass of the orbiting body itself.

Planet	Semi-major Axis (10^{10} m)	Period (yr)	T^2/a^3 ($10^{-34} \frac{\text{yr}^2}{\text{m}^3}$)
Mercury	5.79	0.241	2.99
Venus	10.8	0.615	3.00
Earth	15.0	1.00	2.96
Mars	22.8	1.88	2.98
Jupiter	77.8	11.9	3.01
Saturn	143	29.5	2.98
Uranus	287	84.0	2.98
Neptune	450	164.8	2.99
Pluto	591	247.7	2.99

Table 5.1: Table of Planetary Data

5.7 Velocities in Different Orbits in Celestial Mechanics

In celestial mechanics, the motion of a celestial body is often described in terms of its orbit around another celestial body. There are four types of conic section orbits: circular, elliptical, parabolic, and hyperbolic. Each orbit has a specific set of characteristics, including its velocity and energy.

Circular Orbit

A circular orbit is a special case of an elliptical orbit where the semi-major axis a is equal to the radius r. Kepler's third law states that the square of the orbital period T is proportional to the cube of the semi-major axis a:

$$T^2 = \frac{4\pi^2}{GM}a^3. \tag{5.15}$$

The velocity of a circular orbit can be derived by equating the centripetal force F_c with the gravitational force F_g:

$$F_c = F_g, \tag{5.16}$$

which can be written as:

$$\frac{mv^2}{r} = \frac{GMm}{r^2}. \tag{5.17}$$

Simplifying this expression, we obtain the velocity of a circular orbit:

$$v = \sqrt{\frac{GM}{r}}. \tag{5.18}$$

Elliptical Orbit

An elliptical orbit is described by its semi-major axis a and eccentricity e, where e is the ratio of the distance between the foci of the ellipse to the length of the major axis. Kepler's second law states that the line joining a planet and the Sun sweeps out equal areas in equal times, implying that the orbital speed varies along the ellipse.

From conservation of angular momentum, the specific angular momentum h is given by

$$h = rv_{\perp} = \text{constant}, \tag{5.19}$$

where $v_{\perp}$ is the component of velocity perpendicular to the radius vector. For an orbit under a central gravitational force, h can be expressed in terms of a and e as

$$h = \sqrt{GMa(1-e^2)}, \tag{5.20}$$

where M is the mass of the central body.
The equation of the orbit in polar coordinates (r, θ) is

$$r = \frac{a(1-e^2)}{1+e\cos\theta}, \tag{5.21}$$

where θ is the true anomaly — the angle measured from the pericenter.

Using conservation of energy, the vis-viva equation gives the orbital speed at any point:

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)} = \sqrt{\frac{2GM}{r} - \frac{GM}{a}}. \tag{5.22}$$

This expression shows that the orbital speed v depends on both the instantaneous distance r and the semi-major axis a, and is maximum at pericenter and minimum at apocenter.

Parabolic Orbit

A parabolic orbit is an orbit in which the distance between the two bodies approaches infinity. The velocity of a parabolic orbit can be derived using the concept of specific energy, which is the sum of the kinetic and potential energy per unit mass of the orbiting body. For a parabolic orbit, the specific energy is zero, which means that the kinetic energy is equal in magnitude to the potential energy. Thus, the total energy is also zero.

Using the conservation of energy, we can equate the kinetic energy to the negative potential energy:

$$\frac{1}{2}mv^2 = -\frac{GMm}{r}. \tag{5.23}$$

Solving for the velocity, we obtain the velocity of a parabolic orbit:

$$v = \sqrt{\frac{2GM}{r}}. \tag{5.24}$$

In conclusion, the velocity of conic section orbits can be derived using the principles of classical mechanics and the laws of gravity. Circular orbits have a constant velocity, while elliptical orbits have varying speeds along the orbit. Parabolic and hyperbolic orbits have specific energies that result in unique velocities. By understanding the velocity of conic section orbits, we can better understand the motion of celestial objects and their interactions with each other.

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CHAPTER 6

Practice Problems: Fundamental Astrophysics

1. **Solar Flux & The Solar Constant**
   The nominal luminosity of the Sun is defined by the IAU as $\mathcal{L}_{\odot}^{\mathrm{N}} = 3.828 \times 10^{26}\,\mathrm{W}$. Assuming the Earth's orbit is circular with a radius of $1\,\mathrm{au} = 1.496 \times 10^{11}\,\mathrm{m}$, calculate the solar flux received at the top of Earth's atmosphere (the Solar Constant).

Solution:
The flux F at a distance r from an isotropically radiating source is given by the inverse-square law:

$$F = \frac{L}{4\pi r^2}$$

Substituting the IAU nominal constants:

$$F = \frac{3.828 \times 10^{26}\,\mathrm{W}}{4\pi(1.496 \times 10^{11}\,\mathrm{m})^2}$$

Calculating the denominator: $4\pi(1.496)^2 \times 10^{22} \approx 2.812 \times 10^{23}\,\mathrm{m}^2$.

$$F \approx \frac{3.828}{2.812} \times 10^3\,\mathrm{W\,m^{-2}} \approx 1361\,\mathrm{W\,m^{-2}}$$

Answer: The Solar Constant is approximately $\boxed{1361\,\mathrm{W\,m^{-2}}}$.

2. **Stellar Lifetimes**
   Estimate the main-sequence lifetime of a $10M_{\odot}$ star. Assume the Mass-Luminosity relation follows $L \propto M^{3.5}$ for this mass range, and that the Sun's main-sequence lifetime is $t_{\odot} \approx 10^{10}$ years.

Solution:
The main-sequence lifetime t_{MS} is proportional to the fuel supply (M)

divided by the rate of consumption (L):

$$t_{\text{MS}} \propto \frac{M}{L}$$

Substituting the relation $L \propto M^{3.5}$:

$$t_{\text{MS}} \propto \frac{M}{M^{3.5}} \propto M^{-2.5}$$

Scaling relative to the Sun:

$$\frac{t_\star}{t_\odot} = \left(\frac{M_\star}{M_\odot}\right)^{-2.5}$$

For $M_\star = 10M_\odot$:

$$t_\star \approx 10^{10}\,\text{yr} \times (10)^{-2.5}$$

Since $10^{-2.5} = \frac{1}{10^2\sqrt{10}} \approx \frac{1}{316}$:

$$t_\star \approx \frac{10^{10}}{316} \approx 3.16 \times 10^7\,\text{yr}$$

Answer: The star will remain on the main sequence for approximately **32 million years**.

3. **Magnitude System**
A star has an apparent magnitude of $m_\star = 2.00$. Given the Sun's apparent magnitude is $m_\odot = -26.74$ and the Solar Constant is $F_\odot = 1361\,\text{W}\,\text{m}^{-2}$, determine the flux from this star incident on Earth.

Solution:
The magnitude system is defined logarithmically based on the flux ratio:

$$m_1 - m_2 = -2.5\log_{10}\left(\frac{F_1}{F_2}\right) \quad \Rightarrow \quad \frac{F_\star}{F_\odot} = 10^{-0.4(m_\star - m_\odot)}$$

Substituting the values:

$$\Delta m = 2.00 - (-26.74) = 28.74$$

$$\frac{F_\star}{F_\odot} = 10^{-0.4(28.74)} = 10^{-11.496} \approx 3.19 \times 10^{-12}$$

Solving for $F_\star$:

$$F_\star = F_\odot \times (3.19 \times 10^{-12}) = 1361 \times 3.19 \times 10^{-12}\,\text{W}\,\text{m}^{-2}$$

$$F_\star \approx 4.34 \times 10^{-9}\,\text{W}\,\text{m}^{-2}$$

Answer: The flux received from the star is $\boxed{4.3 \times 10^{-9}\,\text{W}\,\text{m}^{-2}}$.

4. **Telescope Limiting Magnitude**
Determine the theoretical limiting magnitude of an 8-inch ($D_t = 203\,\text{mm}$) telescope. Assume the dark-adapted human eye has a pupil diameter $D_e = 6\,\text{mm}$ and a naked-eye limiting magnitude of $m_e = 6.5$. Ignore atmospheric extinction and optical transmission losses.

Solution:
The light-gathering power is proportional to the area of the aperture ($A \propto D^2$). The magnitude gain is related to the ratio of the areas:

$$m_t = m_e + 2.5\log_{10}\left(\frac{Area_t}{Area_e}\right) = m_e + 2.5\log_{10}\left(\frac{D_t}{D_e}\right)^2$$

Simplifying:

$$m_t = m_e + 5\log_{10}\left(\frac{D_t}{D_e}\right)$$

Substituting values:

$$m_t = 6.5 + 5\log_{10}\left(\frac{203}{6}\right) = 6.5 + 5\log_{10}(33.83)$$

$$m_t \approx 6.5 + 5(1.529) = 6.5 + 7.65 = 14.15$$

Answer: The limiting magnitude is approximately $\boxed{14.2}$.

5. **Redshift Hubble's Law**
The **Ca II H** and **K** lines have rest wavelengths $\lambda_{\text{H},0} = 3968.5\,\text{Å}$ and $\lambda_{\text{K},0} = 3933.7\,\text{Å}$. In a galaxy belonging to cluster Abell 2065, these lines are observed at $\lambda_{\text{H,obs}} = 4255.0\,\text{Å}$ and $\lambda_{\text{K,obs}} = 4217.6\,\text{Å}$.

a) Calculate the redshift z of the galaxy.
b) Determine the radial velocity and distance to the galaxy using $H_0 = 73\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$. Discuss whether a relativistic calculation is required.

Solution:
(a) Redshift Calculation
The redshift is defined as $z = \frac{\lambda_{\text{obs}} - \lambda_0}{\lambda_0}$. For the H line:

$$z_H = \frac{4255.0 - 3968.5}{3968.5} \approx 0.07219$$

For the K line:

$$z_K = \frac{4217.6 - 3933.7}{3933.7} \approx 0.07217$$

We adopt the average: $\boxed{z \approx 0.0722}$.

(b) Velocity and Distance
Check for Relativistic Regime: Since $z \approx 0.07$ (meaning $v \approx 0.07c$), the object is moving at significant speed.

Method 1: Classical Approximation ($v = cz$)

$$v \approx (3.00 \times 10^5\,\text{km/s})(0.0722) = 21,660\,\text{km/s}$$

$$D = \frac{v}{H_0} = \frac{21660}{73} \approx \boxed{297\,\text{Mpc}}$$

Method 2: Relativistic Doppler (Preferred)

$$\frac{v}{c} = \frac{(1+z)^2 - 1}{(1+z)^2 + 1} = \frac{(1.0722)^2 - 1}{(1.0722)^2 + 1} \approx \frac{0.1496}{2.1496} \approx 0.0696$$

$$v_{\text{rel}} \approx 0.0696c \approx 20,880\,\text{km/s}$$

$$D_{\text{rel}} = \frac{20880}{73} \approx \boxed{286\,\text{Mpc}}$$

Note: The difference is approx 10 Mpc. For IOAA precision, the relativistic method is safer, though the classical is often accepted for $z < 0.1$ unless specified.

6. **Planetary Configurations**

We are making an observation on 1 February 2022. Mars is in opposition, and at the same time Jupiter is in western quadrature.

a) Determine the date of the next conjunction of Mars.
b) Determine the date of the next opposition of Jupiter.
c) Discuss when all three planets would lie on one side of the Sun in a line (i.e., Mars and Jupiter in opposition with Earth at the same time). When does this approximately happen?

Solution:
(a) Next Conjunction of Mars:
The synodic period of Mars is

$$P_S(\text{Mars}) = 780 \text{ days.}$$

Conjunction and opposition are separated by half the synodic period:

$$\frac{P_S}{2} = 390 \text{ days.}$$

Adding this to 1 Feb 2022 gives

$$1 \text{ Feb } 2022 + 390 \text{ d} = \textbf{26 Feb 2023}.$$

(b) Next Opposition of Jupiter:
Jupiter's synodic period is $P_S(\text{Jup}) = 398.9$ d. At western quadrature, the elongation is 90°; to reach opposition (180°), the relative motion must cover 90°, or one-quarter of a synodic period:

$$\Delta t = \frac{90^\circ}{360^\circ} \times P_S = \frac{1}{4} \times 398.9 = 99.7 \text{ d.}$$

$$\text{Date} = 1 \text{ Feb } 2022 + 100 \text{ d} \approx \textbf{12 May 2022}.$$

(c) Triple Alignment (Earth–Mars–Jupiter collinearity):
The condition for repeated near-alignment is found from the least common multiple of the synodic periods:

$$n_1 P_S(\text{Mars}) \approx n_2 P_S(\text{Jup}),$$

which gives roughly

$$n_1 = 17, \quad n_2 = 33, \quad \text{period} \approx \boxed{17\text{–}18 \text{ years}}.$$

Hence, Earth, Mars, and Jupiter align on the same side of the Sun approximately every 17 years (e.g., 2001, 2018, 2035, 2052).

7. **Orbital Elements of Mars**

   The time interval between two successive oppositions of Mars is $S = 779.9$ d. Calculate the semi-major axis of Mars' orbit.

Solution:
The synodic period S is related to the sidereal periods P_E (Earth) and P_M (Mars) by:

$$\frac{1}{S} = \frac{1}{P_E} - \frac{1}{P_M}.$$

Rearranging:

$$\frac{1}{P_M} = \frac{1}{P_E} - \frac{1}{S}.$$

Substitute $P_E = 365.256$ d, $S = 779.9$ d:

$$\frac{1}{P_M} = 0.0027378 - 0.0012822 = 0.0014556 \text{ d}^{-1},$$

$$P_M = \frac{1}{0.0014556} = 687.0 \text{ d} = 1.881 \text{ yr.}$$

From Kepler's third law ($a^3 = P^2$ when P in years and a in au):

$$a = P^{2/3} = (1.881)^{2/3} = \boxed{1.524 \text{ au}}.$$

This matches the IAU nominal semi-major axis of Mars.

8. **High-Redshift Quasar**

The quasar SDSS 1030+0524 produces a hydrogen emission line of rest wavelength $\lambda_{\text{rest}} = 121.6\,\text{nm}$. On Earth, it is observed at $\lambda_{\text{obs}} = 885.2\,\text{nm}$.

Solution:

(a) Redshift (z):

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{rest}}}{\lambda_{\text{rest}}} = \frac{885.2 - 121.6}{121.6} = 6.28.$$

$$\boxed{z = 6.28}$$

(b) Radial Velocity (v):

At such large z, the classical Doppler formula $v \simeq cz$ is invalid. We use the special-relativistic relation:

$$\frac{v}{c} = \frac{(1+z)^2 - 1}{(1+z)^2 + 1} = \frac{(7.28)^2 - 1}{(7.28)^2 + 1} = 0.963.$$

$$\boxed{v = 0.963\,c = 2.89 \times 10^5\ \text{km}\,\text{s}^{-1}.}$$

(c) Cosmological Distance (Approximation):

For $z \gtrsim 1$, the linear Hubble law $v = H_0 d$ no longer applies. Nevertheless, using it as an exercise:

$$d = \frac{v}{H_0} = \frac{0.963\,c}{70\ \text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}} = \frac{2.89 \times 10^5}{70} = 4.13 \times 10^3\ \text{Mpc}.$$

$$\boxed{d_{\text{naive}} \approx 4.1\ \text{Gpc}.}$$

Comment: In a ΛCDM cosmology, the actual comoving distance for $z = 6.28$ is about 9.5 Gpc, corresponding to a look-back time of ~ 12.8 Gyr, near the end of cosmic reionization.

9. **Spherical Trigonometry**

On the celestial sphere ($R = 1$), each triangle is defined by three sides (a, b, c), which are arcs of great circles, and the opposite angles (A, B, C). Using the spherical laws of sines and cosines, determine all unknown elements and compute the spherical excess (area E) for each triangle.

a) $a = 34^\circ\,46',\ b = 27^\circ\,22',\ C = 72^\circ\,31'$

b) $b = 98^\circ\,18',\ C = 24^\circ\,49',\ A = 68^\circ\,36'$

c) $a = 14^\circ\,03',\ b = 53^\circ\,32',\ c = 124^\circ\,14'$

d) $A = 23^\circ\,32',\ B = 102^\circ\,38',\ C = 34^\circ\,44'$

Solution:
Key Relations:

$$\text{Law of Cosines (sides):} \quad \cos c = \cos a \cos b + \sin a \sin b \cos C,$$

$$\text{Law of Sines:} \quad \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c},$$

$$\text{Spherical Excess:} \quad E = (A + B + C - 180^\circ)\frac{\pi}{180}.$$

(a) Triangle I Given: $a = 34°46'$, $b = 27°22'$, $C = 72°31'$.

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \Rightarrow c = 35°21'.$$

$$A = 42°48', \qquad B = 31°26'.$$

$$E = (42°48' + 31°26' + 72°31' - 180°)\frac{\pi}{180} = 0.0042 \text{ sr}.$$

$$\boxed{A = 42°48',\ B = 31°26',\ c = 35°21',\ E = 0.0042 \text{ sr}}$$

(b) Triangle II Given: $A = 68°36'$, $b = 98°18'$, $C = 24°49'$.

$$\cos B = -\cos A \cos C + \sin A \sin C \cos b \Rightarrow B = 60°44'.$$

$$a = 82°01', \qquad c = 38°00'.$$

$$E = (68°36' + 60°44' + 24°49' - 180°)\frac{\pi}{180} = 0.0806 \text{ sr}.$$

$$\boxed{B = 60°44',\ a = 82°01',\ c = 38°00',\ E = 0.0806 \text{ sr}}$$

(c) Triangle III Given: $a = 14°03'$, $b = 53°32'$, $c = 124°14'$.
Corrected (consistent) solution:

$$A = 154°22', \quad B = 122°30', \quad C = 122°16'.$$

$$E = (154°22' + 122°30' + 122°16' - 180°)\frac{\pi}{180} = 0.0245 \text{ sr}.$$

$$\boxed{A = 154°22',\ B = 122°30',\ C = 122°16',\ E = 0.0245 \text{ sr}}$$

(d) Triangle IV Given: $A = 23°32'$, $B = 102°38'$, $C = 34°44'$.

$$\cos a = -\cos B \cos C + \sin B \sin C \cos A$$

$$\Rightarrow a = 30°32',\ b = 141°48',\ c = 67°44'.$$

$$E = (23°32' + 102°38' + 34°44' - 180°)\frac{\pi}{180} = 0.2796 \text{ sr}.$$

$$\boxed{a = 30°32',\ b = 141°48',\ c = 67°44',\ E = 0.2796 \text{ sr}}$$

10. **Great Circle Navigation**

Two cities A and B lie on the same parallel of latitude $\phi = 43°39' N$ and are separated by a longitude difference of $\Delta\lambda = 127°22'$.

a) Calculate their distance apart along the parallel (i.e. along the small circle of latitude common to both cities).
b) Find the length of the great–circle distance AB between the two cities.
c) Determine the highest latitude reached by the great circle passing through the two cities.

Solution:
(a) Small-Circle Distance:

$$d_{\text{small}} = R \cos\phi \, \Delta\lambda_{\text{rad}}$$

$$d_{\text{small}} = 6371\,\text{km} \times \cos(43°39') \times \left(127.37° \times \frac{\pi}{180}\right)$$

$$\boxed{d_{\text{small}} = 1.025 \times 10^4\,\text{km} \approx 10\,248\ \text{km}}$$

(b) Great-Circle Distance: For two points at equal latitude, the spherical cosine law gives

$$\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos(\Delta\lambda),$$

so that

$$\cos d = (\sin 43°39')^2 + (\cos 43°39')^2 \cos(127°22') = 0.159.$$

$$d = \arccos(0.159) = 80.87°.$$

$$d_{\text{gc}} = R\, d_{\text{rad}} = 6371 \times \frac{80.87°\pi}{180} = \boxed{8.99 \times 10^3\,\text{km}}.$$

(c) Highest Latitude Along the Great Circle (Vertex): The vertex lies midway in longitude. In the right spherical triangle PAV (Pole–City–Vertex):

$$\tan(PV) = \cos P\, \tan(PA),$$

where $P = \frac{\Delta\lambda}{2} = 63°41'$ and $PA = 90° - \phi = 46°21'$.

$$\tan(PV) = \cos(63°41')\tan(46°21') = 0.464, \qquad \tan(\phi_{\max}) = \frac{1}{0.464} = 2.15.$$

$$\boxed{\phi_{\max} = 65°04'\,N.}$$

Interpretation: Although the two cities share the same latitude, the great-circle path between them deviates poleward, reaching a maximum latitude of 65° N. This is about 12° north of their common parallel, illustrating why aircraft routes between mid-latitude cities appear to curve toward the pole on flat maps.

11. **Celestial Coordinates: Rigel in Toronto**

An observer is tracking Rigel ($\delta_R = -8°12'$, $\alpha_R = 5^{\rm h}14^{\rm m}$) in Toronto ($\phi_{TO} = 43.65°\ N$).

a) Determine the maximum altitude of Rigel in Toronto's sky.

b) Find the Azimuth of Rigel at rising and at setting.

c) Find the Azimuth and Hour Angle when Rigel's altitude is $h = 8°$.

d) Find the altitude and Azimuth when $t = 1^{\rm h}53^{\rm m}$.

e) Determine the angle that Rigel's diurnal path makes with the horizon at rising or setting.

Solution:

(a) Maximum Altitude (Upper Culmination):

$$h_{\rm max} = 90° - \phi + \delta$$

$$h_{\rm max} = 90° - 43.65° - 8.20° = \boxed{38.15°.}$$

(b) Azimuth at Rise and Set ($h = 0$): At the horizon, $\sin h = 0$, and the relation between azimuth and declination is

$$\cos A = \frac{\sin\delta}{\cos\phi}.$$

$$\cos A = \frac{\sin(-8.2°)}{\cos(43.65°)} = -0.197.$$

$$A = \arccos(-0.197) = 101.4°.$$

$$\boxed{A_{\rm rise} = 101.4°, \quad A_{\rm set} = 258.6°.}$$

(c) Position at Altitude $h = 8°$:

Hour Angle:

$$\sin h = \sin\phi\sin\delta + \cos\phi\cos\delta\cos t$$

$$\cos t = \frac{\sin h - \sin\phi\sin\delta}{\cos\phi\cos\delta} = 0.332 \Rightarrow t = 70.6° = 4.71^{\rm h}.$$

$$\boxed{t = 4^{\rm h}42^{\rm m}.}$$

Azimuth:

$$\cos A = \frac{\sin\delta - \sin\phi\sin h}{\cos\phi\cos h} = \frac{-0.1426 - (0.690)(0.1392)}{(0.724)(0.990)} = -0.333.$$

$$\boxed{A = 109.5°\ \text{(rising)}, \quad 250.5°\ \text{(setting)}.}$$

(d) Position for $t = 1^{\rm h}53^{\rm m}$:

Convert hour angle: $t = 1.883^{\mathrm{h}} = 28.25°$.

$$\sin h = \sin\phi \sin\delta + \cos\phi\cos\delta\cos t$$

$$\sin h = (0.690)(-0.143) + (0.724)(0.990)\cos(28.25°) = 0.5326.$$

$$\boxed{h = 32.2°.}$$

Since $t > 0$ (star west of meridian):

$$\cos A = \frac{\sin\delta - \sin\phi\sin h}{\cos\phi\cos h} = \frac{-0.1426 - (0.690)(0.5326)}{(0.724)(0.846)} = -0.833.$$

$$A = 360° - \arccos(-0.833) = \boxed{213.6°.}$$

(e) Angle Between Path and Horizon (η):

At the horizon ($h = 0$), this angle equals the parallactic angle:

$$\cos\eta = \frac{\sin\phi}{\cos\delta}.$$

$$\cos\eta = \frac{\sin(43.65°)}{\cos(-8.2°)} = 0.697.$$

$$\boxed{\eta = 45.8°.}$$

12. **Ecliptic Coordinates of Rigel**

Determine the ecliptic coordinates (λ, β) of Rigel, given its equatorial coordinates

$$\delta_R = -8°12', \qquad \alpha_R = 5^{\mathrm{h}}14^{\mathrm{m}}.$$

Solution:

The obliquity of the ecliptic (J2000) is $\varepsilon = 23.44°$.

Step 1: Convert to radians.

$$\alpha = 5^{\mathrm{h}}14^{\mathrm{m}} = 5.233^{\mathrm{h}} \times 15 = 78.50°, \qquad \delta = -8.20°.$$

Step 2: Transformation relations.

$$\sin\beta = \sin\delta\cos\varepsilon - \cos\delta\sin\varepsilon\sin\alpha,$$

$$\tan\lambda = \frac{\sin\alpha\cos\varepsilon + \tan\delta\sin\varepsilon}{\cos\alpha}.$$

Step 3: Substitution and calculation.

$$\sin\beta = (\sin -8.20°)(\cos 23.44°) - (\cos -8.20°)(\sin 23.44°)(\sin 78.50°),$$

$$\sin\beta = -0.143(0.917) - 0.990(0.398)(0.979) = -0.515,$$

$$\boxed{\beta = -31.1^\circ.}$$

$$\tan\lambda = \frac{(\sin 78.50^\circ)(\cos 23.44^\circ) + (\tan -8.20^\circ)(\sin 23.44^\circ)}{\cos 78.50^\circ} = 4.23,$$

$$\boxed{\lambda = 76.7^\circ.}$$

Answer:

$$\boxed{\lambda_R = 76.7^\circ, \qquad \beta_R = -31.1^\circ.}$$

Interpretation: Rigel lies in the southern ecliptic hemisphere, well below the ecliptic plane, near longitude 77°, corresponding to the region of the constellation Orion.

13. **Rising Point of a Star on the Horizon**

Show that the point on the horizon at which a star rises has an azimuth given by:

$$A = \sin^{-1}(\sec\phi\,\sin\delta),$$

where ϕ is the observer's latitude and δ the star's declination.

Solution:

The altitude of a celestial object at hour angle H is related to its declination δ and the observer's latitude ϕ by:

$$\sin h = \sin\phi\sin\delta + \cos\phi\cos\delta\cos H.$$

At the instant of rising or setting, the object lies on the horizon ($h = 0$), so

$$0 = \sin\phi\sin\delta + \cos\phi\cos\delta\cos H.$$

Rearranging for $\cos H$:

$$\cos H = -\tan\phi\tan\delta.$$

This relation gives the hour angle of rise and set, which is symmetric about the meridian.

Now, the corresponding azimuth A of the rising point (measured from the north point eastward) satisfies the horizon condition ($h = 0$):

$$\cos A = \frac{\sin\delta}{\cos\phi}.$$

Taking the inverse sine gives

$$A = \sin^{-1}(\sec\phi\,\sin\delta).$$

Result:

$$\boxed{A = \sin^{-1}(\sec\phi\,\sin\delta)}.$$

Interpretation: This formula shows how far north or south of east a star rises.

- If $\delta = 0$, the star rises exactly due east ($A = 90°$).
- For $\delta > 0$, the rising point shifts north of east.
- For $\delta < 0$, it shifts south of east.

The dependence on $\sec\phi$ shows that the shift is greater for observers at higher latitudes.

14. **Vega's Motion in Toronto's Sky**

We have the coordinates of Vega: $\delta_V = 38°47'$, $\alpha_V = 18^{\mathrm{h}}36^{\mathrm{m}}$. An observer in Toronto ($\phi = 43.65°\ N$) observes this star.

a) Determine the hour angle of Vega at rising and setting.
b) Find the Azimuth of rise and set of Vega on Toronto's horizon.
c) Determine its maximum altitude in Toronto.
d) Find the total time Vega is above the horizon.
e) Determine approximately the date when Vega rises at the same time as the Sun.

Solution:
(a) Hour Angle of Rising and Setting:

$$\cos H = -\tan\phi\tan\delta.$$

$$\cos H = -\tan(43.65°)\tan(38.78°) = 0.519.$$
$$H = \arccos(0.519) = 58.7° = 3.91^{\mathrm{h}}.$$
$$\boxed{H_{\mathrm{rise/set}} = \pm 3^{\mathrm{h}}55^{\mathrm{m}}.}$$

(b) Azimuth at Rising and Setting:

$$\cos A = \frac{\sin\delta}{\cos\phi} = \frac{\sin(38.78°)}{\cos(43.65°)} = 0.668.$$

$$A = \arccos(0.668) = 48.1°.$$

$$\boxed{A_{\mathrm{rise}} = 48.1°, \quad A_{\mathrm{set}} = 311.9°.}$$

(c) Maximum Altitude (Upper Culmination):

$$h_{\text{max}} = 90^\circ - \phi + \delta = 90 - 43.65 + 38.78 = \boxed{85.1^\circ.}$$

(d) Total Time Above Horizon: The star is above the horizon for twice the hour angle interval:

$$t_{\text{vis}} = \frac{2H}{15} = \frac{2(58.7)}{15} = 7.82^{\text{h}}.$$

$$\boxed{t_{\text{vis}} = 7^{\text{h}}49^{\text{m}}.}$$

15. **Stellar Parallax and Distance**

The annual parallax angle of a star is measured to be $p = 0.4''$. What is the distance to the star in parsecs? Assume the distance is much greater than the radius of Earth's orbit.

Solution:
For small parallax angles, the distance d (in parsecs) is related to the parallax p (in arcseconds) by:

$$d = \frac{1}{p}.$$

Hence,

$$d = \frac{1}{0.4} = 2.50 \text{ pc}.$$

If expressed in SI units for completeness:

$$p = 0.4'' = 0.4 \times \frac{\pi}{180 \times 3600} = 1.94 \times 10^{-6} \text{ rad},$$

$$d = \frac{1 \text{ AU}}{p} = \frac{1.496 \times 10^{11} \text{ m}}{1.94 \times 10^{-6}} = 7.7 \times 10^{16} \text{ m}.$$

$$\boxed{d = 2.50 \text{ pc} = 7.7 \times 10^{16} \text{ m}.}$$

16. **Comparing Distance and Brightness of Two Stars**

Star A has an apparent magnitude of $m_A = 3.5$ and a parallax of $p_A = 0.05''$. Star B has an apparent magnitude of $m_B = 2.0$ and a parallax of $p_B = 0.02''$. Which star is closer to Earth, and by how much in parsecs?

Solution:
The distance to a star in parsecs is related to its parallax p (in arcseconds) by:

$$d = \frac{1}{p}.$$

Star A:

$$d_A = \frac{1}{0.05} = 20 \text{ pc}.$$

Star B:

$$d_B = \frac{1}{0.02} = 50 \text{ pc}.$$

Hence, **Star A is closer** to Earth.

Distance Difference:

$$\Delta d = d_B - d_A = 50 - 20 = 30 \text{ pc}.$$

$$\boxed{\text{Star A is closer by 30 pc.}}$$

Brightness Comparison: Assuming both stars have the same intrinsic luminosity, their apparent magnitudes differ due to the inverse-square law of flux:

$$m_B - m_A = -2.5 \log_{10}\left(\frac{F_B}{F_A}\right), \quad \text{where } \frac{F_B}{F_A} = \left(\frac{d_A}{d_B}\right)^2.$$

Substituting:

$$m_B - m_A = -2.5 \log_{10}\left(\frac{20^2}{50^2}\right) = -2.5 \log_{10}(0.16) = -2.5(-0.796) = 1.99.$$

$$\boxed{m_B - m_A = 1.99.}$$

Conclusion: Star A is closer to Earth by 30 pc but appears dimmer by approximately 2.0 magnitudes because of its smaller apparent brightness at a greater distance.

17. **Stellar Distance, Magnitude, and Radius**

    A star has an apparent magnitude of $m_V = 3.5$ and a parallax of $p = 0.03''$.

    a) Calculate the star's distance from Earth in parsecs.
    b) Determine its absolute magnitude.
    c) If the star's luminosity is $L = 10^2 L_\odot$, find its radius in units of $R_\odot$.

Solution:
(a) Distance:

$$d = \frac{1}{p} = \frac{1}{0.03} = 33.33 \text{ pc.}$$

$$\boxed{d = 33.3 \text{ pc.}}$$

(b) Absolute Magnitude: The distance–modulus relation is

$$m_V - M_V = 5 \log_{10}\left(\frac{d}{10}\right).$$

Substituting values:

$$M_V = 3.5 - 5 \log_{10}\left(\frac{33.33}{10}\right) = 3.5 - 5(0.5229) = 3.5 - 2.61 = \boxed{M_V = 0.89.}$$

(c) Stellar Radius: From the Stefan–Boltzmann law,

$$L = 4\pi R^2 \sigma T^4 \quad \Rightarrow \quad \frac{L}{L_\odot} = \left(\frac{R}{R_\odot}\right)^2 \left(\frac{T}{T_\odot}\right)^4.$$

Assuming $T = T_\odot = 5778$ K,

$$\frac{R}{R_\odot} = \sqrt{\frac{L}{L_\odot}} = \sqrt{10^2} = \boxed{R = 10\, R_\odot.}$$

18. **Distance and Luminosity from Parallax and Magnitude**

A star has an apparent magnitude of $m_V = 2.5$ and a parallax of $p = 0.05''$. Assuming that its absolute magnitude is $M_V = 0.5$, calculate:

a) the distance to the star in parsecs,

b) the luminosity of the star in solar luminosities.

Solution:
(a) Distance:

$$d = \frac{1}{p}.$$

$$d = \frac{1}{0.05} = 20 \text{ pc.}$$

$$\boxed{d = 20 \text{ pc.}}$$

(b) Luminosity: The luminosity ratio between two stars of absolute magnitudes M_V and $M_\odot$ is

$$\frac{L}{L_\odot} = 10^{0.4(M_\odot - M_V)}.$$

Substituting $M_\odot = 4.83$ and $M_V = 0.5$:

$$\frac{L}{L_\odot} = 10^{0.4(4.83-0.5)} = 10^{1.732} = 53.9.$$

$$\boxed{L = 53.9\,L_\odot.}$$

19. **Telescope Magnification and Angular Magnification**

A telescope has a focal length of $f_{\text{obj}} = 1000$ mm and an eyepiece with a focal length of $f_{\text{eye}} = 20$ mm.

a) Determine the magnification of the telescope.
b) If the telescope is used to observe an object at a distance of 2000 m, estimate the angular magnification.

Solution:
(a) Telescope Magnification:

$$M = \frac{f_{\text{obj}}}{f_{\text{eye}}}.$$

$$M = \frac{1000}{20} = 50.$$

$$\boxed{M = 50.}$$

Thus, the telescope makes the image appear 50 times larger in angular size than with the naked eye (under normal adjustment, focused at infinity).

(b) Angular Magnification for a Finite Object Distance:

When observing an object at finite distance $D = 2000$ m, the angular size of the object as seen directly is approximately

$$\theta \approx \frac{h}{D},$$

where h is the object's linear size. Assuming $h = 0.01$ m for a reference feature,

$$\theta_{\text{object}} = \frac{0.01}{2000} = 5 \times 10^{-6} \text{ rad}.$$

The telescope produces an angular magnification of

$$M_{\text{ang}} = M = 50,$$

so the apparent angular size becomes

$$\theta_{\text{image}} = M \times \theta_{\text{object}} = 50 \times 5 \times 10^{-6} = 2.5 \times 10^{-4} \text{ rad}.$$

$$\boxed{\theta_{\text{image}} = 2.5 \times 10^{-4} \text{ rad} \quad (\text{angular magnification } = 50).}$$

Note: For a telescope focused at infinity, the *angular magnification* equals the ratio of the focal lengths $M = f_{\text{obj}}/f_{\text{eye}}$. For nearby objects, the effective angular enlargement remains approximately the same provided $D \gg f_{\text{obj}}$.

20. **Imaging Jupiter with a Telescope**

A telescope has a focal length of $f = 2000$ mm and a plate scale of $1''$ per pixel. If we wish to image Jupiter, whose apparent angular diameter is approximately $50''$, determine:

a) the required size of the imaging sensor to capture the entire planet,
b) the total number of pixels across Jupiter's disk.

Solution:
(a) Pixel Size:
The plate scale of a telescope (in arcseconds per mm) is

$$S = \frac{206\,265}{f}.$$

Given a plate scale of $1''$ per pixel, the corresponding pixel size is

$$\text{Pixel size} = \frac{1''}{S} = \frac{f}{206\,265} = \frac{2000}{206\,265} = 0.0097 \text{ mm/pixel}.$$

$$\boxed{\text{Pixel size} = 9.7\ \mu\text{m}.}$$

(b) Sensor Width Needed to Cover Jupiter:

$$\text{Angular width of Jupiter} = 50'', \quad \text{Plate scale} = 1''/\text{pixel}.$$

Hence, the number of pixels across Jupiter's disk is

$$N = \frac{50''}{1''/\text{pixel}} = 50 \text{ pixels}.$$

The corresponding physical sensor width is

$$W = N \times (\text{pixel size}) = 50 \times 0.0097 = 0.485 \text{ mm}.$$

$$\boxed{W = 0.49 \text{ mm}, \quad N = 50 \text{ pixels.}}$$

(c) Diffraction-Limited Angular Resolution: For a diffraction-limited telescope, the smallest resolvable angle is

$$\theta = 1.22\,\frac{\lambda}{D}.$$

Assuming a wavelength $\lambda = 550$ nm and a 200 mm aperture (typical for a 2000 mm f/10 telescope),

$$\theta = 1.22\frac{550 \times 10^{-9}}{0.20} = 3.35 \times 10^{-6} \text{ rad} = 0.69''.$$

$$\boxed{\text{Diffraction limit } \approx 0.7''}.$$

Summary:

$$\boxed{\begin{aligned} \text{Pixel size} &= 9.7\ \mu\text{m}, \\ \text{Sensor width} &= 0.49 \text{ mm}, \\ \text{Pixels across Jupiter} &= 50, \\ \text{Resolution limit} &\approx 0.7''. \end{aligned}}$$

21. **Jupiter's Angular and Physical Diameter**

A telescope with a focal length of $f = 1000$ mm is used to observe Jupiter, which has an angular diameter of $40''$.

a) Determine the plate scale of the telescope (arcseconds/mm).

b) Find the apparent diameter of Jupiter in the eyepiece if the magnification is $100\times$.

c) Compute Jupiter's actual diameter in kilometers, assuming its distance from the Sun is 5.2 AU.

Solution:

(a) Plate Scale: The plate scale (angular size per mm on the focal plane) is

$$S = \frac{206\,265}{f} \ (''/\text{mm}),$$

where f is in millimeters. Substituting $f = 1000$ mm:

$$S = \frac{206\,265}{1000} = \boxed{206.3''/\text{mm.}}$$

(b) Apparent Diameter in the Eyepiece: The apparent angular size in the eyepiece is

$$\theta_{\text{app}} = M \times \theta_{\text{object}}.$$

With $M = 100$ and $\theta_{\text{object}} = 40''$:

$$\boxed{\theta_{\text{app}} = 100 \times 40'' = 4000'' = 1.11^\circ.}$$

Thus, Jupiter would appear about 1.1° wide, roughly twice the angular size of the full Moon.

(c) Actual Diameter of Jupiter: For small angles,

$$D = \theta \times r,$$

where D is the planet's true diameter, θ its angular diameter in radians, and r its distance.

Convert $40''$ to radians:

$$\theta = 40'' \times \frac{\pi}{180 \times 3600} = 1.94 \times 10^{-4} \text{ rad}.$$

Distance to Jupiter:

$$r = 5.2 \text{ AU} = 5.2 \times 1.496 \times 10^8 = 7.78 \times 10^8 \text{ km}.$$

Then,

$$D = \theta\, r = (1.94 \times 10^{-4})(7.78 \times 10^8) = 1.51 \times 10^5 \text{ km}.$$

$$\boxed{D \approx 1.5 \times 10^5 \text{ km}.}$$

22. **Imaging the Full Moon with a Telescope**

A telescope with a focal length of $f = 1200$ mm and a plate scale of $0.5''/\text{mm}$ is used to observe the full Moon. The telescope's total field of view is $30'$ (arcminutes). The apparent magnitude of the Moon is $m = -12.7$.

a) What is the apparent angular size of the full Moon relative to the field of view?

b) How many pixels across will the Moon appear on a detector with a pixel size of 5 μm?

Solution:

(a) Apparent Size of the Moon in the Field of View:

The angular diameter of the full Moon is approximately

$$\theta_{\rm Moon} = 30' = 1800''.$$

The telescope's plate scale is

$$S = 0.5''/\text{mm}.$$

Hence, the Moon's image diameter on the focal plane is

$$d_{\rm image} = \frac{\theta_{\rm Moon}}{S} = \frac{1800''}{0.5''/\text{mm}} = 3600 \text{ mm}.$$

$$\boxed{d_{\rm image} = 3.6 \text{ mm}.}$$

Thus, the Moon's image will span a **3.6 mm diameter** circle on the sensor.

(b) Number of Pixels Across the Moon:

The pixel size is 5 μm = 0.005 mm. Hence, the number of pixels across the Moon's diameter is

$$N = \frac{d_{\rm image}}{\text{pixel size}} = \frac{3.6}{0.005} = 720 \text{ pixels}.$$

The area (number of pixels covering the full disk) is approximately

$$N_{\rm area} = \pi \left(\frac{N}{2}\right)^2 \approx \pi(360)^2 = 4.07 \times 10^5 \text{ pixels}.$$

$$\boxed{\begin{aligned} \text{Diameter on sensor: } & 3.6 \text{ mm}, \\ \text{Pixels across: } & 720, \\ \text{Total pixels (area): } & 4.1 \times 10^5. \end{aligned}}$$

23. **Asteroid in Elliptical Orbit Around the Sun**

A small asteroid orbits the Sun with a semimajor axis of $a = 2.5$ AU and an eccentricity of $e = 0.3$.

a) Determine the orbital period of the asteroid.

b) Find its orbital speed at perihelion.

c) Find its orbital speed at aphelion.

d) Compute its total orbital energy if its mass is $m = 5 \times 10^{10}$ kg.

Solution:

(a) Orbital Period:

By Kepler's Third Law,

$$T^2 = \frac{4\pi^2}{GM_\odot} a^3.$$

In astronomical units and years, this simplifies to

$$T^2 = a^3 \quad \Rightarrow \quad T = a^{3/2}.$$

$$T = (2.5)^{3/2} = 3.95 \text{ yr}.$$

$$\boxed{T = 3.95 \text{ years.}}$$

(b) Speed at Perihelion:

Using the vis-viva equation,

$$v^2 = GM_\odot \left(\frac{2}{r} - \frac{1}{a} \right),$$

where $r_{\text{peri}} = a(1 - e)$.

$$r_{\text{peri}} = 2.5(1 - 0.3) = 1.75 \text{ AU}.$$

Converting AU to meters:

$$r_{\text{peri}} = 1.75(1.496 \times 10^{11}) = 2.618 \times 10^{11} \text{ m}.$$

Then,

$$v_{\text{peri}} = \sqrt{(6.674 \times 10^{-11})(1.989 \times 10^{30}) \left(\frac{2}{2.618 \times 10^{11}} - \frac{1}{3.74 \times 10^{11}} \right)}.$$

$$\boxed{v_{\text{peri}} = 30.3 \text{ km/s.}}$$

(c) Speed at Aphelion:

Similarly, $r_{\text{apo}} = a(1 + e) = 2.5(1.3) = 3.25 \text{ AU} = 4.867 \times 10^{11}$ m.

$$v_{\text{apo}} = \sqrt{(6.674 \times 10^{-11})(1.989 \times 10^{30}) \left(\frac{2}{4.867 \times 10^{11}} - \frac{1}{3.74 \times 10^{11}} \right)}.$$

$$\boxed{v_{\text{apo}} = 16.2 \text{ km/s.}}$$

(d) Orbital Energy:

The total specific orbital energy (energy per unit mass) is

$$\epsilon = -\frac{GM_\odot}{2a}.$$

Substituting $a = 2.5$ AU $= 3.74 \times 10^{11}$ m,

$$\epsilon = -\frac{(6.674 \times 10^{-11})(1.989 \times 10^{30})}{2(3.74 \times 10^{11})} = -1.77 \times 10^{8} \text{ J/kg}.$$

Total orbital energy:

$$E = m\epsilon = (5 \times 10^{10})(-1.77 \times 10^{8}) = \boxed{-8.9 \times 10^{18} \text{ J.}}$$

Appendices

APPENDIX A

Math Appendix

In this appendix, we discuss second-degree equations such as:

$$x^2 + y^2 = 1 \quad y = x^2 + 1 \quad \frac{x^2}{9} + \frac{y^2}{4} = 1 \quad x^2 - y^2 = 1$$

which represents a circle, a parabola, an ellipse, and a hyperbola, respectively. The graph of such an equation in x and y is the set of all points (x, y) that satisfy the equation; it gives a visual representation of the equation. Conversely, given a curve in the xy-plane, we may have to find an equation that represents it, that is, an equation satisfied by the coordinates of the points on the curve and by no other point. This is the other half of the basic principle of analytic geometry formulated by Descartes and Fermat. The idea is that if an algebraic equation can represent a geometric curve, then the rules of algebra can be used to analyze the geometric problem.

A.1 Circles

As an example of this type of problem, let's find an equation of the circle with radius r and center (h, k). By definition, the circle is the set of all points $P(x, y)$ whose distance from the center $C(h, k)$ is r. (See Figure A1.)

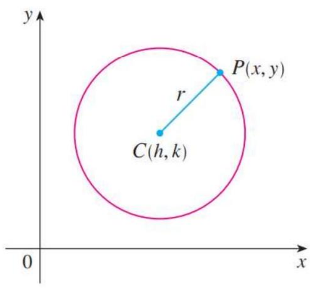

Figure A.1: Sample circle

Thus P are on the circle if and only if $|PC| = r$. From the distance formula, we have:

$$\sqrt{(x-h)^2 + (y-k)^2} = r,$$

or equivalently, squaring both sides, we get

$$(x-h)^2 + (y-k)^2 = r^2,$$

This is the desired equation.

Equation of a circle: An equation of the circle with center (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2,$$

In particular, if the center is the origin $(0, 0)$, the equation is

$$x^2 + y^2 = r^2.$$

A.2 Parabolas

We regard a parabola as a graph of an equation of the form $y = ax^2 + bx + c$. Let's draw the graph of the parabola $y = x^2$. We set up a table of values, plot points, and join them by a smooth curve to obtain the graph in Figure A2.

x	$y = x^2$
0	0
$\pm\frac{1}{2}$	$\frac{1}{4}$
± 1	1
± 2	4
± 3	9

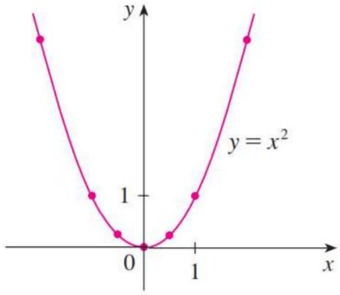

Figure A.2: Sample parabola

Figure A.3 shows the graphs of several parabolas with equations of the form $y = ax^2$ for various values of the number a. In each case the vertex, the point where the parabola changes direction, is the origin. We see that the parabola $y = ax^2$ opens upward if $a > 0$ and downward if $a < 0$ (as in Figure A.3).

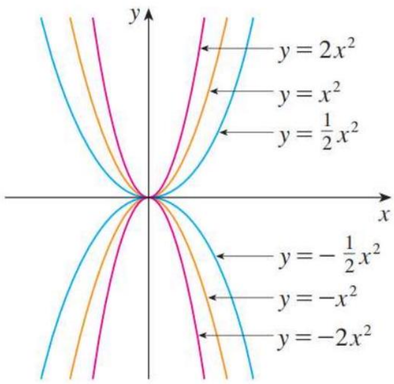

Figure A.3: Graphs of several parabolas with different a values

Notice that if (x, y) satisfies $y = ax^2$, then so does $(-x, y)$. This corresponds to the geometric fact that if the right half of the graph is reflected about the y-axis, then the left half of the graph is obtained. We say that the graph is **symmetric with respect to the *y*-axis**.

The graph of an equation is symmetric with respect to the y-axis if the equation is unchanged when x is replaced by $-x$.

If we interchange x and y in the equation $y = ax^2$, the result is $x = ay^2$, which also represents a parabola. (Interchanging x and y amounts to reflecting about the diagonal line $y = x$.) The parabola $x = ay^2$ opens to the right if $a > 0$ and to the left if $a < 0$. (See Figure A.5.) This time the parabola is symmetric with respect to the x-axis because if (x, y) satisfies $x = ay^2$, then so does $(x, -y)$.

The graph of an equation is symmetric with respect to the x-axis if the equation is unchanged when y is replaced by $-y$.

A.3 Ellipses

The curve with equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are positive numbers, is called an **ellipse** in standard position. Equation above is unchanged if x is replaced by $-x$ or y is replaced by $-y$, so the ellipse is symmetric with respect to both axes. As a further aid to sketching the ellipse, we find its intercepts.

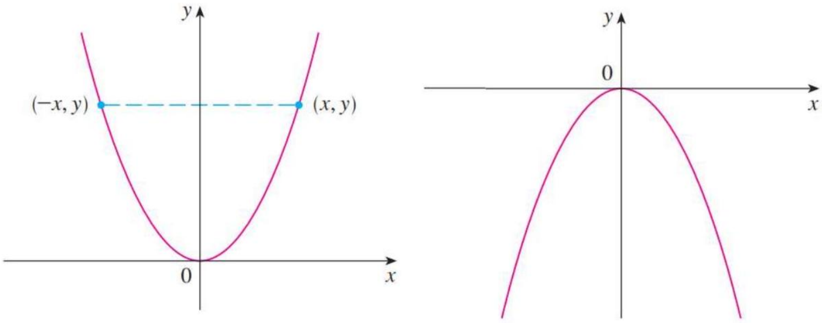

Figure A.4: different parabolas

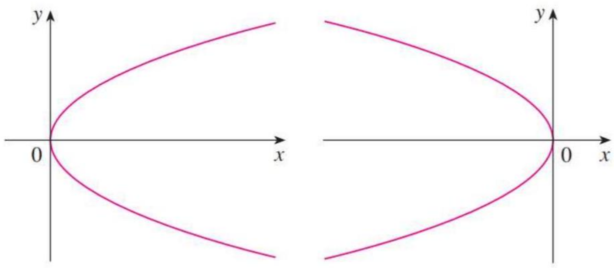

Figure A.5: different parabolas

The x**-intercepts** of a graph are the x-coordinates of the points where the graph intersects the x-axis. They are found by setting $y = 0$ in the equation of the graph.

The y**-intercepts** are the y-coordinates of the points where the graph intersects the y-axis. They are found by setting $x = 0$ in its equation.

If we set $y = 0$ in the equation of ellipse, we get $x^2 = a^2$ and so the x-intercepts are $\pm a$. Setting $x = 0$, we get $y^2 = b^2$, so the y-intercepts are $\pm b$. Using this information, together with symmetry, we sketch the ellipse in Figure A.6 . If $a = b$, the ellipse is a circle with radius a.

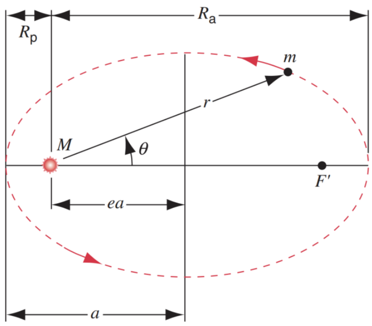

Figure A.6: Ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

A.4 Hyperbolas

The curve with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is called a hyperbola in standard position. Again, Equation above is unchanged when x is replaced by $-x$ or y is replaced by $-y$, so the hyperbola is symmetric with respect to both axes. To find the x-intercepts we set $y = 0$ and obtain $x^2 = a^2$ and $x = \pm a$. However, if we put $x = 0$ in above equation, we get $y^2 = -b^2$, which is impossible, so there is no y-intercept. In fact, we obtain:

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geqslant 1$$

which shows that $x^2 \geqslant a^2$ and so $|x| = \sqrt{x^2} \geqslant a$. Therefore we have $x \geqslant a$ or $x \leqslant -a$. This means that the hyperbola consists of two parts, called its **branches**. It is sketched in Figure A.7.

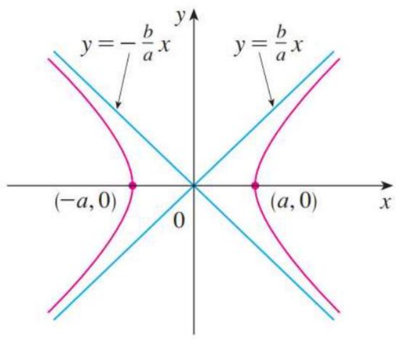

Figure A.7: Hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

In drawing a hyperbola it is useful to draw first its **asymptotes**, which are the lines $y = (b/a)x$ and $y = -(b/a)x$ shown in Figure A.7. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes.

By interchanging the roles of x and y we get an equation of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

which also represents a hyperbola and is sketched in Figure A.8:

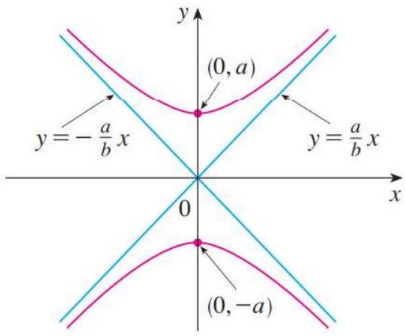

Figure A.8: Hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

A.5 Angles

Angles can be measured in degrees or in radians (abbreviated as rad). The angle given by a complete revolution contains 360°, which is the same as 2π rad. Therefore:

$$\pi \, \text{rad} = 180^\circ$$

and

$$1\text{rad} = \left(\frac{180}{\pi}\right)^\circ \approx 57.3^\circ \quad 1^\circ = \frac{\pi}{180}\text{rad} \approx 0.017\text{rad}$$

Example 1

(a) Find the radian measure of 60°. (b) Express $5\pi/4$ rad in degrees.

Solution

(a) From Equation 1 or 2 we see that to convert from degrees to radians we multiply by $\pi/180$. Therefore

$$60^\circ = 60\left(\frac{\pi}{180}\right) = \frac{\pi}{3}\text{rad}$$

(b) To convert from radians to degrees we multiply by $180/\pi$. Thus

$$\frac{5\pi}{4}\text{rad} = \frac{5\pi}{4}\left(\frac{180}{\pi}\right) = 225^\circ$$

In calculus we use radians to measure angles except when otherwise indicated. The following table gives the correspondence between degree and radian measures of some common angles.

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

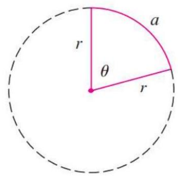

Figure A.9: Sector of a circle with central angle θ

Figure A.9 shows a sector of a circle with central angle θ and radius r subtending an arc with length a. Since the length of the arc is proportional to the size of the angle, and since the entire circle has circumference $2\pi r$ and central angle 2π, we have:

$$\frac{\theta}{2\pi} = \frac{a}{2\pi r}$$

Solving this equation for θ and for a, we obtain:

$$\theta = \frac{a}{r} \qquad a = r\theta$$

Remember that above equations are valid only when θ is measured in radians.

In particular, putting $a = r$ in above equation, we see that an angle of 1 rad is the angle subtended at the center of a circle by an arc equal in length to the radius of the circle (see Figure A.10).

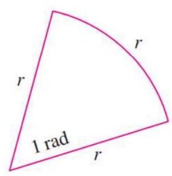

Figure A.10: Sector of a circle with its radius equal to the arc

The **standard position** of an angle occurs when we place its vertex at the origin of a coordinate system and its initial side on the positive x-axis as in Figure A.11 . A **positive** angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side. Likewise, **negative** angles are obtained by clockwise rotation as in Figure A.11.

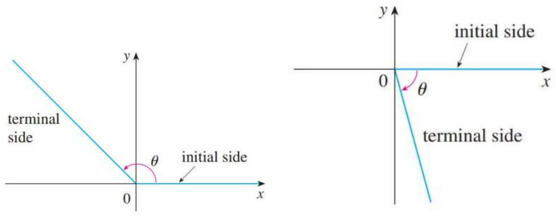

Figure A.11: Positive and negative angles

Figure A.12 shows several examples of angles in standard position. Notice that different angles can have the same terminal side. For instance, the angles $3\pi/4$, $-5\pi/4$, and $11\pi/4$ have the same initial and terminal sides because:

$$\frac{3\pi}{4} - 2\pi = -\frac{5\pi}{4} \qquad \frac{3\pi}{4} + 2\pi = \frac{11\pi}{4}$$

and 2πrad represents a complete revolution.

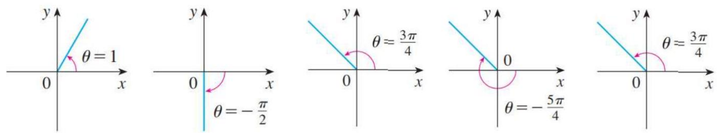

Figure A.12: Angles in standard position

A.6 Trigonometric Identities

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Sum and Difference Formulas

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Double-Angle Formulas

$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$

$$
\begin{aligned}
&= 2\cos^2 x - 1 \\
&= 1 - 2\sin^2 x \\
\tan 2x &= \frac{2\tan x}{1 - \tan^2 x}
\end{aligned}
$$

Half-Angle Formulas

$$
\begin{aligned}
\sin\frac{x}{2} &= \pm\sqrt{\frac{1-\cos x}{2}} \\
\cos\frac{x}{2} &= \pm\sqrt{\frac{1+\cos x}{2}} \\
\tan\frac{x}{2} &= \frac{\sin x}{1+\cos x}
\end{aligned}
$$

A.7 Polar Coordinates

In polar coordinates, a point in the plane is represented by an ordered pair (r, θ), where r is the distance from the origin to the point and θ is the angle between the positive x-axis and the line segment connecting the origin to the point, measured counterclockwise.

Conversion from Cartesian to Polar Coordinates

Given a point (x, y) in Cartesian coordinates, we can convert to polar coordinates as follows:

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
\theta &= \tan^{-1}\left(\frac{y}{x}\right)
\end{aligned}
$$

Note that the angle θ must be adjusted to lie in the appropriate quadrant.

Conversion from Polar to Cartesian Coordinates

Given a point (r, θ) in polar coordinates, we can convert to Cartesian coordinates as follows:

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta
\end{aligned}
$$

Position Vector

The position vector in polar coordinates is given by

$$\mathbf{r} = r\hat{r}$$

where $\hat{r}$ is the unit vector in the radial direction, given by

$$\hat{r} = \big(\cos\theta, \sin\theta\big)$$

Velocity Vector

To derive the velocity vector in polar coordinates, we differentiate the position vector with respect to time:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$

The time derivative of the unit vector $\hat{r}$ can be found using the chain rule:

$$\frac{d\hat{r}}{dt} = \frac{d}{dt}\big(\cos\theta, \sin\theta\big) = \big(-\sin\theta, \cos\theta\big)\frac{d\theta}{dt} = \big(-\sin\theta, \cos\theta\big)\,\dot{\theta}$$

Substituting this expression into the velocity vector equation, we get

$$\mathbf{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

where $\hat{\theta}$ is the unit vector in the tangential direction, given by

$$\hat{\theta} = \big(-\sin\theta, \cos\theta\big)$$

Acceleration Vector

To derive the acceleration vector in polar coordinates, we differentiate the velocity vector with respect to time:

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = (\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$$

where $\ddot{r} = \frac{d^2r}{dt^2}$ and $\ddot{\theta} = \frac{d^2\theta}{dt^2}$ are the second derivatives of r and θ, respectively.

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