CANADIAN ASTRONOMY AND ASTROPHYSICS OLYMPIAD

# 2025 Problems

8 problems  Total: 240 points

## Problem 1  Properties of Sirius and its Companion (20 points)

Find the physical characteristics of the star Sirius ($\alpha$ Canis Majoris) and of its companion from the following observational data:

- The visual magnitude of Sirius is $V_1 = -1.46$ and its colour index is $(B-V)_1 = 0.00$.
- The companion has apparent magnitude $V_2 = +8.50$ and colour index $(B-V)_2 = +0.15$.
- The parallax of Sirius is $\pi = 0.375''$.
- The companion revolves around Sirius with a period $P = 50$ years.
- The angular semi-major axis of the orbit is $a'' = 7.60''$.
- The ratio of the distances of the two stars from their common centre of mass is $a_2 : a_1 = 2.3 : 1$.
- The absolute visual magnitude of the Sun is $M_{\odot V} = +4.77$.

## Problem 2  Average Density of a B0V Star (15 points)

Calculate the average density of a main-sequence star of spectral class B0V. The star has an effective temperature $T = 30\,000$ K and a mass of approximately $15\,M_\odot$.

Use known relationships between stellar parameters to estimate its radius, and then determine the density. The average density is

$$\rho = \frac{3M}{4\pi R^3},$$

where $M$ is the mass of the star and $R$ its radius.

## Problem 3 Migration Paths of the Ruby-throated Hummingbird (25 points)

For centuries people have watched bird migrations, using them to predict the seasons and to understand natural navigation. The Ruby-throated Hummingbird (*Archilochus colubris*) is one of the most remarkable migratory species, travelling thousands of kilometres each year between North and Central America.

The hummingbird breeds in Toronto (43.7° N, 79.4° W) and migrates to its wintering grounds in San José, Costa Rica (9.9° N, 84.1° W). Because of the Coriolis effect and the prevailing winds, the path taken by the birds is not a straight line but can be approximated by an **ellipse** on the Earth's surface, with one focus $F_1$ at Kansas City, USA (39.1° N, 94.6° W). The second focus is to be determined.

The major axis of this elliptical route is 50°.

(a) Determine the coordinates of the second focus $F_2$.

(b) Compute the eccentricity of the migration path, determine the coordinates of the apsis points (the points of the ellipse closest to and farthest from the foci), and find the conjugate points at the ends of the minor axis.

Note: treat the Earth as a perfect sphere of radius 6371 km.

## Problem 4 Properties of Kuiper Belt Objects (30 points)

The fraction of sunlight reflected by an object is called its **albedo**; the rest is absorbed and re-emitted as thermal radiation. Kuiper Belt Objects (KBOs) orbit beyond Neptune. They are so far away that most appear as point sources that reflect and emit radiation.

(a) An astronomer wants to measure the reflected flux from these bodies. In which wavelength range should the astronomer observe, and why?

(b) Astronomer X has measured the reflected flux $F_R$ (energy per unit area per unit time at a ground-based detector) when the object is at opposition. Derive an expression for $F_R$ in terms of the object's distance from the Sun $D$, the Sun's luminosity $L$, the object's radius $R$ (assume it is spherical) and its albedo $A$.

(c) Astronomer Y has measured the thermal flux $F_E$ emitted by the object. Derive the corresponding expression for $F_E$, using the result of part (b).

(d) At opposition the observed KBOs have a proper motion $\mu = 3$ arcseconds per hour relative to the distant stars. This motion is solely due to parallax. Use it to estimate $D$ in astronomical units.

(e) From the results of parts (c) and (d), determine the wavelength at which astronomer Y observed the emitted radiation.

(f) Suppose both astronomers know $D$. (i) Can astronomer X determine $R$ alone? (ii) Can astronomer Y determine $R$ alone?

## Problem 5 Gamma-Ray Bursts in Distant Galaxies (30 points)

Gamma-ray bursts (GRBs) are among the most energetic explosions in the universe, originating from the collapse of massive stars (long GRBs) or the merger of compact objects such as neutron stars (short GRBs). They last from a few milliseconds to several minutes and produce photons with energies around 1 MeV.

To be observed from the Earth, a GRB must give a flux of at least 50 photons per second per square centimetre. The luminosity of these explosions is approximately $10^{49}$ erg/s. In the Euclidean model used here, the explosion is a point source emitting in a conical jet with an opening angle of $10^\circ$. About one GRB per week is detected on the Earth.

(a) What fraction of the sky can observe these GRBs?

(b) Estimate the distance to the farthest GRB that can be observed from the Earth.

(c) What is the approximate frequency of GRBs in a distant galaxy?

(d) If the jet opening angle were $5^\circ$ instead of $10^\circ$, how many GRBs would be observable from the Earth per week?

## Problem 6 Comet Ejection from the Oort Cloud (30 points)

A star of mass $0.5\,M_\odot$ enters the Oort Cloud with a velocity of 30 km/s. Assume the Oort Cloud is a spherical shell of radius $10^5$ AU containing a large number of comets.

(a) How close (in astronomical units) must the star pass to one of the comets for the comet to be ejected from the Oort Cloud? State any assumptions you need.

(b) If the age of the Solar System is $4 \times 10^9$ years, what fraction of the comets in the Oort Cloud would be ejected by this mechanism? Assume that the number density of stars in the solar neighbourhood is 0.1 per cubic parsec, that their average mass is $0.5\,M_\odot$ and that their average velocity is also 30 km/s.

(c) If the radius of the Oort Cloud were $10^4$ AU, how would the fraction found in part (b) change?

## Problem 7 Cosmological Redshift (30 points)

When an object moves away from us its light is shifted to longer wavelengths. This is the **redshift**

$$z = \frac{\lambda_{\text{received}} - \lambda_e}{\lambda_e},$$

where $\lambda_{\text{received}}$ is the observed wavelength and $\lambda_e$ the emitted one. Edwin Hubble showed that for distant galaxies $v = H_0 d$, where $H_0 = 75$ km/s/Mpc is the Hubble constant, $d$ the distance of the galaxy and $v$ its velocity. The universe is expanding, and the wavelengths of light are stretched with it.

(a) The size of the universe at time $t$ is $r(t) = a(t)\, r(t_0)$, where $a(t)$ is the **scale factor** and $r(t_0) = r_0$ is the present size. Show that $a = \dfrac{1}{1+z}$.

(b) For a universe in which all particles move much more slowly than light, the mass density depends on $z$. Derive this dependence.

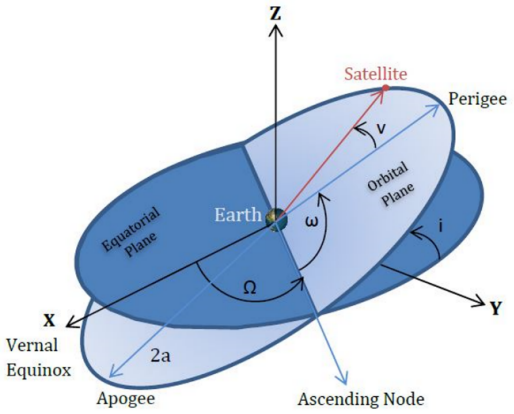
(c) The **critical density** is the density for which the total energy (kinetic plus potential) of cosmic objects is zero. Write the total energy and derive an expression for the critical density.

(d) Compute the critical density in kg/m$^3$.

(e) Express the critical density in solar masses per cubic parsec.

## Problem 8 Satellite Motion in the ECI Frame (60 points)

The ECI (Earth-Centred Inertial) frame is an inertial frame centred on the Earth, with the $x$-axis pointing to the vernal equinox and the $z$-axis to the north celestial pole, so that the $xy$-plane is the plane of the celestial equator. Six parameters, the orbital elements, are needed to describe an orbit completely; they are shown in the figure.




The satellite's orbital elements.

Consider a satellite with orbital elements $i = 0$, $e = 0.15$, $a = 2.5\,R_e$, $\Omega = 0$, $\omega = -30^\circ$ in the ECI frame. Its orbit lies in the plane of the celestial equator; the major axis of the ellipse is rotated clockwise by $30^\circ$ from the $x$-axis, and the centre of the Earth is at one focus. Derive relationships in symbols unless a numerical value is explicitly requested by the verb *calculate*.

We study the motion of this satellite as seen by an observer in Toronto, $\varphi = 43.65^\circ$ N, $l = 79.5^\circ$ W $= -5^{\mathrm{h}}18^{\mathrm{m}}$. It is known that the satellite is at perigee at 15:20 Pacific Time (GMT $-8$) on December 10. The satellite is close enough to the Earth that the observer in Toronto does not see it in the same direction as an observer at the Earth's centre. The Sun's orbit is taken to be circular with radius 1 AU; the Sun is a distant object, the satellite is not.

The *mean Sun* moves with the same period as the real Sun but along the celestial equator; both start from the vernal equinox and meet again at the autumnal equinox. Local time is $LT = H_{MS} + 12^{\mathrm{h}}$, where $H_{MS}$ is the hour angle of the mean Sun. Zone time follows from $GMT = LT - l$; for Eastern Time, $ZT = GMT - 5^{\mathrm{h}}$.

(a) Calculate the declination and right ascension of the Sun on this day.

(b) From now on, let $t = 0$ be the moment the satellite is at perigee. Calculate how long it has been since sunset in Toronto at that moment, to the nearest minute.

(c) At $t = 0$, give the position vectors of the satellite and of the observer in the ECI frame, in symbols and numerically.

(d) Write the observer's position vector in the ECI frame as a function of time.

(e) Write the satellite's position vector in the ECI frame as a function of time, using Kepler's equation.

Since the eccentricity is small, we may work to first order in $e$: $f(e) \approx f(0) + f'(0)\,e$.

(f) Show that to first order the eccentric anomaly is $E = \Omega t - e \sin \Omega t$, where $\Omega = 2\pi/T$ and $T$ is the period.

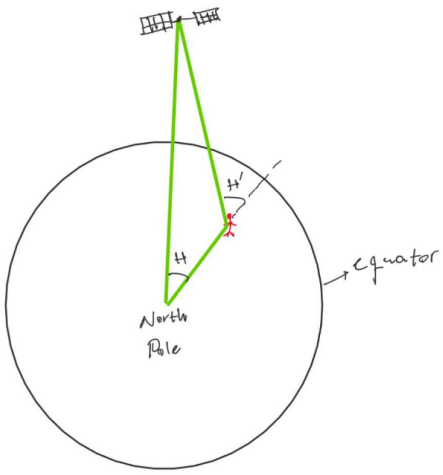
(g) Show that to first order the true anomaly is $\theta = E + 2e \sin E$, and write it as $\theta = p(t) + e\,q(t)$. Explain why $p$ is linear in time and $q$ is periodic with period $T$, and find both.

(h) Write the satellite's position vector as a function of time to first order in $e$.

(i) Formulate the condition for the rising and setting of the satellite in terms of the two position vectors.

(j) Calculate the date and time of the satellite's first rise and set as seen from Toronto, in Eastern Time.

(k) Calculate the satellite's hour angle $H'$ at its first rise and set. (It is not the same as the hour angle $H$ seen from the centre of the Earth.)




(l) At rise and set, the satellite's tangential velocity relative to the observer makes an angle with the horizon. Report this angle for the first rise and set.

(m) Find the Doppler shift of the satellite's signal as a function of time between rise and set. Is there a moment when it is zero? If so, give the date and time in Eastern Time.

(n) Derive the azimuth of the satellite as a function of time. Is there a moment when the rate of change of azimuth becomes zero and reverses (like the retrograde motion of planets)? If so, give the date and time.

(o) Derive the altitude of the satellite as a function of time. At what azimuth does the satellite reach its maximum altitude? Should it not be 180°? What is the maximum altitude?