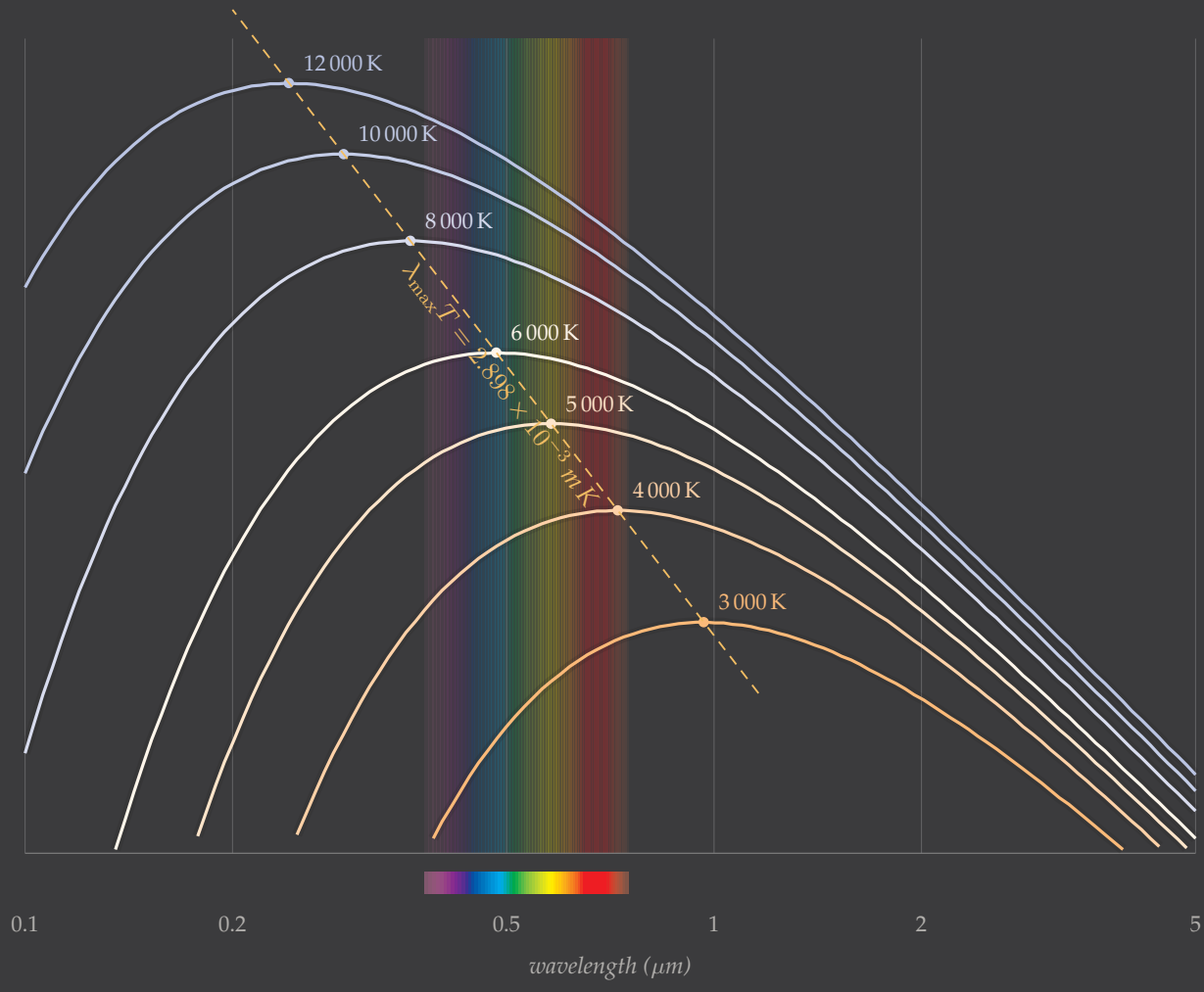
CAAO

Canadian Astronomy and Astrophysics Olympiad

Theory, worked examples and practice problems

Light from stars of 3000 to 12 000 K, and Wien's displacement law



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Preface

This guide is written for students preparing for the Canadian Astronomy and Astrophysics Olympiad (CAAO). Each year the highest-achieving CAAO students are invited to a training programme, from which the team that represents Canada at the International Olympiad on Astronomy and Astrophysics (IOAA) is chosen. Canada has taken part in the IOAA since 2013.

The book assumes a solid background in high-school physics and mathematics and nothing more. Calculus is not needed for the CAAO, although students who go on to the international olympiad will need it, together with physics well beyond the school curriculum. The mathematical tools used here are collected in Appendix A.

How the book is organised

Part I is the theory. Its five chapters follow the order in which the topics are best learned: the light of the stars; telescopes; stars, galaxies and cosmology; the geometry of the sky; and the motion of bodies under gravity. Every new idea is followed at once by a worked example, and many of the examples are short questions from past papers of the international olympiad, chosen because they need nothing beyond the level of the CAAO. Each chapter ends with a table of its formulas.

Part II is practice: a set of problems with full solutions, grouped by chapter, taken or adapted from past IOAA papers and from the first round of the USA olympiad, or written for this book. The source of each past-paper problem is shown beside its title.

How to use it

Astronomy olympiads are won by solving problems, not by reading. Cover the solution of each worked example and try it yourself first. When your answer differs from the one given, find out why before moving on. Check the units and the size of every answer: a star closer than the Sun or a planet moving faster than light means a mistake somewhere. Once you have worked through the book, the complete past papers of the CAAO are the best test of your preparation.

The tables in Appendix C give the constants used throughout. In an examination, always use the values printed on the paper, even when they differ slightly from the ones here.

Resources

No single book covers everything. The following textbooks are recommended for going further; full details are given in the list of references at the end.

1. *Foundations of Astrophysics*, by Barbara Ryden and Bradley Peterson. A clear modern introduction and the best single companion to this guide.
2. *Fundamental Astronomy*, edited by Hannu Karttunen and others. Broad coverage at an introductory level, with good chapters on spherical astronomy and celestial mechanics.
3. *Astronomy: Principles and Practice*, by Archie Roy and David Clarke. The most thorough treatment of spherical astronomy, time and telescopes at this level.
4. *An Introduction to Modern Astrophysics*, by Bradley Carroll and Dale Ostlie. The standard reference for the international olympiad; it needs calculus and university-level physics.
5. *Introduction to Cosmology*, by Barbara Ryden. For students who want to understand the expanding universe properly.

Past papers are as important as textbooks. The papers of the CAAO, the first-round and national papers of the USA Astronomy and Astrophysics Olympiad, and the theoretical and data-analysis papers of the IOAA are all freely available and make an excellent graded sequence of practice, in that order of difficulty.

Finally, go outside. Learn the bright stars and the constellations of each season, follow the Moon through a month and a planet through its retrograde loop. Students who know the real sky find the geometry of Chapter 4 far easier.

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PART I

Theory

CHAPTER 1

Starlight: Distance, Brightness and Colour

Almost everything we know about a star arrives as light. This chapter builds the small toolkit that turns that light into physical quantities: how far away the star is, how much energy it emits, how bright it looks from Earth, and how hot its surface is. Nearly every CAAO paper contains at least one question that uses nothing more than the formulas in this chapter, so it is worth knowing them well.

1.1 Angles on the sky

Astronomers measure the sky in angles. One degree is divided into 60 arcminutes ($1^\circ = 60'$) and one arcminute into 60 arcseconds ($1' = 60''$), so $1^\circ = 3600''$. Most formulas, however, need the angle in radians:

$$1\,\mathrm{rad} = \frac{180^\circ}{\pi} = 57.296^\circ = 206\,265''. \tag{1.1}$$

The number 206 265 appears so often that it is worth memorising.

For a small angle θ (in radians), $\sin\theta \approx \tan\theta \approx \theta$. An object of physical size D at distance d therefore has angular size

$$\theta = \frac{D}{d} \quad \text{(radians)}, \qquad \theta'' = 206\,265\,\frac{D}{d}. \tag{1.2}$$

The approximation is better than one part in a thousand for angles below about 4°.

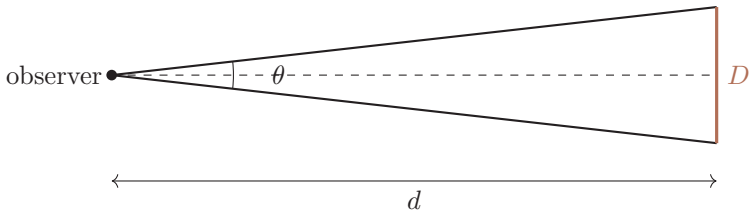


Figure 1.1: The small-angle formula: an object of size D at distance d subtends $\theta = D/d$ radians.

Example 1.1.1 (Angular size of the Moon)**.** The Moon has a diameter of 3474 km and a mean distance of 384 400 km. What is its angular diameter?

Solution.

$$\theta = \frac{3474}{384\,400} = 9.04 \times 10^{-3}\,\text{rad} = 9.04 \times 10^{-3} \times 57.296^\circ = 0.518^\circ \approx 31'.$$

The Sun happens to have almost the same angular diameter (about $32'$), which is why total solar eclipses are possible.

1.2 Parallax and the parsec

Hold a finger at arm's length and look at it with one eye, then the other: the finger jumps against the background. Stars do the same thing as the Earth moves around the Sun. Two observations made six months apart use a baseline equal to the diameter of the Earth's orbit, and a nearby star is seen to shift back and forth against the much more distant background stars. Half of the total shift is called the **parallax angle** p (Figure 1.2).

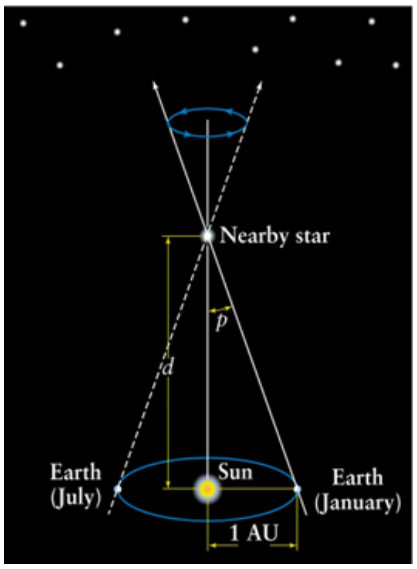


Figure 1.2: The parallax triangle. The baseline is the Earth–Sun distance, 1 AU.

From the right-angled triangle,

$$d = \frac{1\,\text{AU}}{\tan p} \approx \frac{1\,\text{AU}}{p} \qquad (p \text{ in radians}). \tag{1.3}$$

The **parsec** (pc) is defined as the distance at which 1 AU subtends exactly one arcsecond:

$$1\,\text{pc} = 206\,265\,\text{AU} = 3.086 \times 10^{16}\,\text{m} = 3.26 \text{ light-years}.$$

With this unit the parallax formula becomes as simple as a formula can be:

$$d\,[\mathrm{pc}] = \frac{1}{p\,['']}$$

The larger the distance, the smaller the angle. Even the nearest star has a parallax smaller than one arcsecond, which is why stellar parallax was not measured until 1838.

Example 1.2.1 (The nearest star)**.** Proxima Centauri has a parallax of $0.768''$. Find its distance in parsecs, light-years and astronomical units.

Solution. $d = 1/0.768 = 1.30\,\mathrm{pc}$. In light-years this is $1.30 \times 3.26 = 4.25\,\mathrm{ly}$, and in astronomical units $1.30 \times 206\,265 = 2.69 \times 10^5\,\mathrm{AU}$.

1.3 Stars as blackbodies

A hot, dense object emits light at all wavelengths with a spectrum that depends only on its temperature. Such an idealised emitter is called a **blackbody**, and stars are a good approximation to one. Figure 1.3 shows the spectrum for three temperatures. Two things happen as the temperature rises: the peak moves to shorter (bluer) wavelengths, and the total energy emitted grows very quickly.

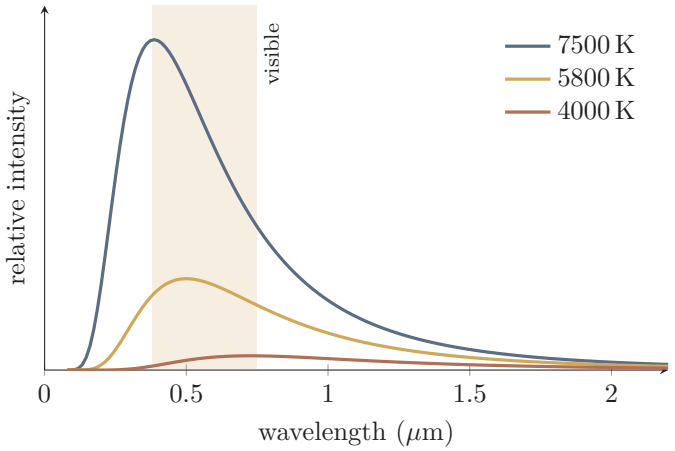


Figure 1.3: Blackbody spectra. A hotter body is brighter at every wavelength and peaks at a shorter wavelength.

Wien's displacement law

The wavelength at which the spectrum peaks is inversely proportional to the temperature:

$$\lambda_{\max}\, T = 2.898 \times 10^{-3}\,\mathrm{m\,K}. \tag{1.4}$$

This is why cool stars look red and hot stars look blue-white.

The Stefan–Boltzmann law

Each square metre of a blackbody at temperature T emits a power σT^4, where $\sigma = 5.670 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$ is the Stefan–Boltzmann constant. A star of radius R has surface area $4\pi R^2$, so its total power output, its **luminosity**, is

$$L = 4\pi R^2 \sigma T^4$$

Luminosity is measured in watts. For the Sun, $L_\odot = 3.828 \times 10^{26}\,\mathrm{W}$, $R_\odot = 6.957 \times 10^{8}\,\mathrm{m}$ and $T_\odot = 5772\,\mathrm{K}$. In solar units the law takes a form that needs no constants:

$$\frac{L}{L_\odot} = \left(\frac{R}{R_\odot}\right)^2 \left(\frac{T}{T_\odot}\right)^4 . \tag{1.5}$$

Example 1.3.1 (Colour and power of two stars [adapted from IOAA 2007, short questions 1.8 and 1.9])**.**

(a) At what wavelength does a star with surface temperature 4000 K emit most strongly?

(b) A star has radius $2.5\,R_\odot$ and surface temperature 7500 K. Find its luminosity in solar units.

Solution.

(a) $\lambda_{\max} = 2.898 \times 10^{-3}/4000 = 7.25 \times 10^{-7}\,\mathrm{m} = 725\,\mathrm{nm}$, just beyond the red end of the visible band. The visible light we see from such a star is the short-wavelength side of its curve, so the star looks orange-red. For comparison the Sun peaks at $2.898 \times 10^{-3}/5772 = 502\,\mathrm{nm}$.

(b) $L/L_\odot = (2.5)^2\,(7500/5772)^4 = 6.25 \times 2.85 = 17.8$.

1.4 Flux and the inverse-square law

Luminosity is what the star emits; **flux** is what we receive. Imagine a sphere of radius d centred on the star (Figure 1.4). If nothing absorbs the light on the way, all of the luminosity passes through that sphere, spread evenly over its area $4\pi d^2$. The flux, the power crossing each square metre, is

$$b = \frac{L}{4\pi d^2}$$

and is measured in $\mathrm{W\,m^{-2}}$. In this book b and F are both used for flux; exam papers use either. Because L is fixed, moving twice as far away makes the star four times fainter. This is the **inverse-square law**.

Example 1.4.1 (The solar constant on Earth and on Mars)**.** Find the flux of sunlight at the Earth ($1\,\mathrm{AU} = 1.496 \times 10^{11}\,\mathrm{m}$) and at Mars (1.524 AU).

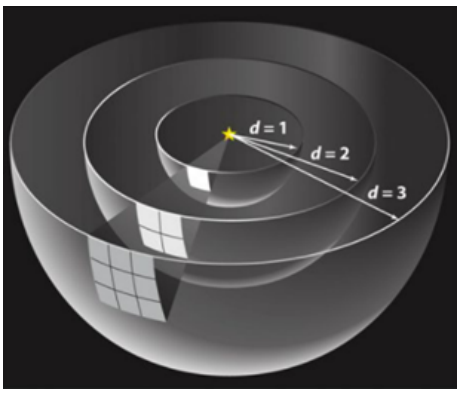


Figure 1.4: The same light spreads over a larger area at a larger distance.

Solution. At the Earth,

$$b_\oplus = \frac{3.828 \times 10^{26}}{4\pi\,(1.496 \times 10^{11})^2} = 1361\,\mathrm{W\,m^{-2}}.$$

This is the *solar constant*. For Mars there is no need to start again: by the inverse-square law $b_{\mathrm{Mars}} = 1361/1.524^2 = 586\,\mathrm{W\,m^{-2}}$.

Example 1.4.2 (A supernova as bright as the Sun [adapted from IOAA 2007, short question 1.11])**.** At maximum a supernova reaches a luminosity of $10^{10}\,L_\odot$. How far away would it have to be to deliver the same flux as the Sun does at the Earth?

Solution. Equal flux means $L/4\pi d^2 = L_\odot/4\pi(1\,\mathrm{AU})^2$, so

$$d = 1\,\mathrm{AU} \times \sqrt{\frac{L}{L_\odot}} = \sqrt{10^{10}}\,\mathrm{AU} = 10^5\,\mathrm{AU} = 0.48\,\mathrm{pc} = 1.6\ \text{light-years}.$$

Because flux depends on the square of the distance, a source 10^{10} times more powerful is equally bright from only 10^5 times farther away.

1.5 The magnitude system

About 2200 years ago Hipparchus sorted the naked-eye stars into six classes: the brightest were 'first magnitude' and the faintest 'sixth magnitude'. The scale therefore runs backwards: *a smaller magnitude means a brighter star*. In the nineteenth century Pogson made the scale exact by defining a difference of 5 magnitudes to be a flux ratio of exactly 100. One magnitude is then a ratio of $100^{1/5} = 2.512$, and for any two objects

$$m_1 - m_2 = -2.5 \log_{10} \frac{b_1}{b_2} \qquad \text{or} \qquad \frac{b_1}{b_2} = 10^{-0.4\,(m_1 - m_2)}$$

The quantity m is the **apparent magnitude**. It describes how bright the object looks from Earth, not how powerful it is. The scale extends in both directions beyond Hipparchus's six classes (Figure 1.5).

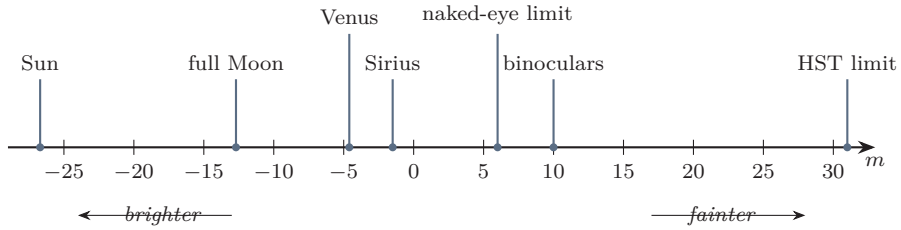


Figure 1.5: The apparent magnitude scale. Each step of 5 magnitudes is a factor of 100 in flux.

Example 1.5.1 (The range of the naked-eye stars [adapted from IOAA 2007, short question 1.15])**.** The brightest star in the night sky has magnitude about -1.5, and the faintest stars visible to the naked eye have magnitude about 6. What is the ratio of their fluxes?

Solution. The difference is 7.5 magnitudes, so the flux ratio is $10^{0.4 \times 7.5} = 10^3 = 1000$. The eye copes with a factor of a thousand between the brightest and the faintest stars.

Adding magnitudes

Magnitudes never add; fluxes do. To combine two or more sources, convert each magnitude to a flux, add the fluxes, and convert back.

Example 1.5.2 (Total magnitude of a binary [IOAA 2010, short problem 1])**.** The two components of a binary star have apparent magnitudes 1.0 and 2.0, and are too close together to be seen separately. Find the magnitude of the pair.

Solution. Take the brighter star as the reference. Then $b_2/b_1 = 10^{-0.4(2.0-1.0)} = 10^{-0.4} = 0.398$, so the total flux is $b_{\text{tot}} = b_1 + b_2 = 1.398\, b_1$. Therefore

$$m_{\text{tot}} = m_1 - 2.5 \log_{10} \frac{b_{\text{tot}}}{b_1} = 1.0 - 2.5 \log_{10}(1.398) = 0.64.$$

A quick check: the answer must be brighter (smaller) than the brighter component alone, and it is. Note that the answer is *not* $1 + 2$.

1.6 Absolute magnitude and the distance modulus

Apparent magnitude mixes two things: how luminous a star is, and how far away it is. To compare stars fairly we imagine moving them all to the same standard distance. The **absolute magnitude** M is the apparent magnitude a star would have at a distance of exactly 10 pc.

Apply the magnitude formula to the same star at its true distance d and at 10 pc. The flux ratio is $(10/d)^2$ by the inverse-square law, so

$$m - M = -2.5 \log_{10} \left(\frac{10}{d} \right)^2 = 5 \log_{10} \frac{d}{10\,\mathrm{pc}},$$

which is usually written as

$$m - M = 5 \log_{10} d - 5 \qquad (d \text{ in parsecs})$$

The difference $m - M$ is called the **distance modulus**. A star at exactly 10 pc has $m = M$; a more distant star has $m > M$.

Since all stars are compared at the same distance, absolute magnitudes compare luminosities directly:

$$M_1 - M_2 = -2.5 \log_{10} \frac{L_1}{L_2}, \qquad \text{so} \qquad \frac{L}{L_\odot} = 10^{\,0.4\,(M_\odot - M)}. \tag{1.6}$$

For the Sun, $m_\odot = -26.74$ and $M_\odot = 4.83$ in the visual (V) band. Summed over all wavelengths (the *bolometric* magnitude) the Sun has $M_{\mathrm{bol},\odot} = 4.74$. Unless a problem tells you otherwise, use the values printed on the exam's table of constants.

Example 1.6.1 (The absolute magnitude of the Sun)**.** Starting from $m_\odot = -26.74$, show that $M_\odot = 4.83$.

Solution. The Sun is at $d = 1\,\mathrm{AU} = 1/206\,265\,\mathrm{pc}$, so $\log_{10} d = -5.314$ and

$$M_\odot = m_\odot - 5 \log_{10} d + 5 = -26.74 + 26.57 + 5 = 4.83.$$

From 10 pc the Sun would be an unremarkable star, just visible to the naked eye.

Example 1.6.2 (A supernova in a nearby galaxy)**.** A Type Ia supernova, which reaches absolute magnitude $M = -19.3$ at maximum, explodes in a galaxy 20 Mpc away. How bright does it appear at maximum? Could it be seen in a small telescope with limiting magnitude 13?

Solution. The distance is 2.0×10^7 pc, so

$$m = M + 5 \log_{10} d - 5 = -19.3 + 5 \log_{10}(2.0 \times 10^7) - 5 = -19.3 + 36.5 - 5 = 12.2.$$

This is just within reach of the telescope. A single exploding star sixty-five million light-years away can be seen with amateur equipment, which is what makes these supernovae such useful distance indicators.

1.7 Colour and temperature

A detector with a filter measures the magnitude in one band of wavelengths. The two most common bands are B (blue, centred near 440 nm) and V (visual, yellow-green, near 550 nm). The difference

$$B - V = m_B - m_V$$

is called a **colour index**. Because it is a difference of two magnitudes of the same star, it does not depend on distance. A hot star emits relatively more blue light, so its B magnitude is *smaller* and $B - V$ is small or negative; a cool star has a large positive $B - V$.

Star	Colour	T (K)	$B - V$
Rigel	blue-white	12 000	-0.03
Vega	white	9600	0.00
Sun	yellow	5772	$+0.65$
Arcturus	orange	4300	$+1.23$
Betelgeuse	red	3500	$+1.85$

Table 1.1: Colour index and surface temperature of some familiar stars.

A convenient empirical relation between colour and temperature is

$$T \approx 4600\,\mathrm{K}\left[\frac{1}{0.92\,(B-V)+1.7}+\frac{1}{0.92\,(B-V)+0.62}\right]. \tag{1.7}$$

You do not need to memorise it; when a problem needs it, it will be given. For the Sun it returns 5780 K.

1.8 Extinction

So far we have assumed that nothing lies between the star and the detector. In reality interstellar dust absorbs and scatters starlight, and so does the Earth's atmosphere. Both effects make a star look *fainter*, so both add a positive number to its magnitude.

Interstellar extinction

If dust dims a star by A magnitudes, the distance modulus becomes

$$\boxed{m - M = 5\log_{10} d - 5 + A}$$

In the plane of the Milky Way, A_V is roughly one magnitude per kiloparsec, though it varies enormously from one direction to another. Dust scatters blue light more than red, so it also makes stars look redder than they really are (*interstellar reddening*).

Example 1.8.1 (What happens if you forget the dust)**.** A star of absolute magnitude $M = -1.0$ lies 2.0 kpc away, behind dust that causes 1.0 magnitude of extinction per kiloparsec. (a) What is its apparent magnitude? (b) What distance would an observer deduce if they ignored extinction?

Solution. (a) $A = 2.0$, so $m = -1.0 + 5\log_{10}(2000) - 5 + 2.0 = 12.5$.

(b) Ignoring A: $5\log_{10} d = m - M + 5 = 18.5$, so $d = 10^{3.7} = 5000\,\mathrm{pc}$. The star would be placed two and a half times too far away. Forgetting extinction always *overestimates* distance.

Atmospheric extinction

Light from a star at the zenith passes through one thickness of atmosphere, called one *air mass*. At zenith distance z the path is longer by a factor

$$X \approx \sec z = \frac{1}{\cos z}, \tag{1.8}$$

which is a good approximation for $z < 70°$ (Figure 1.6). The observed magnitude is

$$m_{\mathrm{obs}} = m_0 + kX, \tag{1.9}$$

where m_0 is the magnitude above the atmosphere and k is the extinction coefficient, typically 0.15 to 0.3 magnitudes per air mass in visible light at a good site.

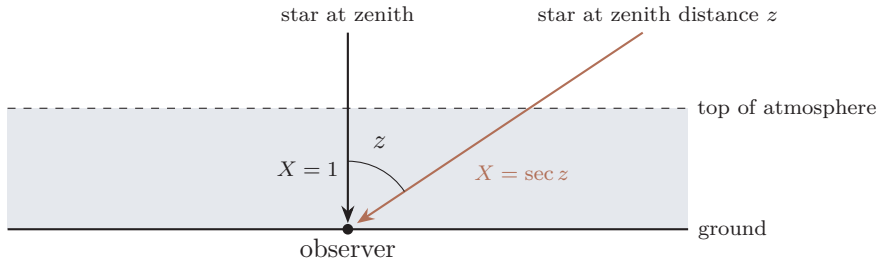


Figure 1.6: Air mass. A star far from the zenith is seen through a longer path of air and is dimmed more.

Example 1.8.2 (Air mass)**.** At a site with $k = 0.25$ magnitudes per air mass, how much fainter does a star appear at altitude 30° than it would at the zenith?

Solution. Altitude 30° means $z = 60°$ and $X = \sec 60° = 2$. The dimming is $kX = 0.50$ mag at this altitude and 0.25 mag at the zenith, so the star is 0.25 magnitudes fainter, a loss of about 21% of its light.

1.9 Surface brightness

A star is a point, but galaxies, nebulae and planets are extended. For them the useful quantity is **surface brightness**: flux per unit solid angle on the sky. In magnitudes,

$$\mu = m + 2.5\log_{10}\Omega \qquad (\text{mag per arcsec}^2), \tag{1.10}$$

where m is the total magnitude and Ω is the area of the object in square arcseconds.

Surface brightness has a remarkable property: *it does not depend on distance.* Move a galaxy twice as far away and its flux falls by a factor of four, but so does the area it covers on the sky, and the ratio is unchanged. (This holds as long as there is no extinction and the distance is not cosmological.) It is why a distant wall looks just as bright as a near one.

Example 1.9.1 (A faint galaxy)**.** A galaxy has total magnitude $m = 10.0$ and appears as an ellipse measuring $3' \times 2'$. Estimate its mean surface brightness.

Solution. The semi-axes are $90''$ and $60''$, so $\Omega = \pi \times 90 \times 60 = 1.70 \times 10^4$ arcsec^2 and

$$\mu = 10.0 + 2.5 \log_{10}(1.70 \times 10^4) = 20.6 \text{ mag arcsec}^{-2}.$$

A dark country sky has a surface brightness of about 21.5 in the same units, so this galaxy stands out only slightly from the background, even though a *star* of magnitude 10 is easy in a small telescope.

1.10 Reflected light: planets and asteroids

Planets, moons and asteroids shine by reflected sunlight. Suppose a body of diameter D and albedo A (the fraction of light it reflects) is at distance r from the Sun, measured in AU, and at distance Δ from the observer. Sunlight arrives at the body with flux $b_\odot / r^2$, where $b_\odot$ is the solar constant. Seen at full phase, the body returns to the observer a flux

$$\frac{b}{b_\odot} \approx \frac{A\, D^2}{4\, r^2\, \Delta^2}, \tag{1.11}$$

with r in AU and D and Δ in the same units as each other. The magnitude then follows from comparing with the Sun, $m - m_\odot = -2.5 \log_{10}(b/b_\odot)$. Equation (1.11) is an estimate: the exact numerical factor depends on how the surface scatters light and on the phase, and CAAO problems that need more detail will say so. The important features are the dependences on D^2 and on $1/(r^2\Delta^2)$.

Example 1.10.1 (How bright is an asteroid?)**.** An asteroid of diameter 100 km and albedo 0.15 is at opposition, 2.8 AU from the Sun and 1.8 AU from the Earth. Estimate its apparent magnitude.

Solution. Put D and Δ in kilometres: $\Delta = 1.8 \times 1.496 \times 10^8 = 2.69 \times 10^8$ km. From Equation (1.11), with $r = 2.8$ as a pure number,

$$\frac{b}{b_\odot} = \frac{0.15 \times 100^2}{4 \times 2.8^2 \times (2.69 \times 10^8)^2} = 6.6 \times 10^{-16},$$

$$m = m_\odot - 2.5 \log_{10} \frac{b}{b_\odot} = -26.74 + 37.95 = 11.2.$$

The asteroid is an easy target for a small telescope but far too faint for the naked eye. Because the flux falls as $1/(r^2\Delta^2)$, small bodies in the outer solar system are extremely hard to find.

1.11 Summary

Small angle	$\theta = D/d$; $1\,\mathrm{rad} = 206\,265''$
Parallax	$d\,[\mathrm{pc}] = 1/p\,['']$
Wien	$\lambda_{\max} T = 2.898 \times 10^{-3}\,\mathrm{m\,K}$
Luminosity	$L = 4\pi R^2 \sigma T^4$
Flux	$b = L/(4\pi d^2)$
Magnitudes	$m_1 - m_2 = -2.5 \log_{10}(b_1/b_2)$
Distance modulus	$m - M = 5 \log_{10} d - 5 + A$
Luminosity from M	$L/L_\odot = 10^{0.4(M_\odot - M)}$
Air mass	$m_{\mathrm{obs}} = m_0 + k \sec z$
Surface brightness	$\mu = m + 2.5 \log_{10} \Omega$

Table 1.2: Formulas of Chapter 1.

CHAPTER 2

Telescopes and Detectors

A telescope does three separate jobs, and it helps to keep them apart in your mind:

- it **collects light**, so that faint objects become detectable;
- it **resolves detail**, separating things that are close together on the sky;
- it **magnifies**, making the image large enough for the eye or the detector to use.

The first two depend on the diameter of the telescope, its **aperture** D. Only the third depends on the eyepiece. This is why astronomers describe a telescope by its aperture and never by its magnification.

2.1 Refractors and reflectors

The main optical element of a telescope, the **objective**, gathers the light and brings it to a focus. In a **refracting telescope** the objective is a lens at the front of a closed tube (Figure 2.1). Galileo turned the first small refractors to the sky in 1609. The largest refractor ever built, at Yerkes Observatory, has a lens 1.02 m across; larger lenses sag under their own weight, because a lens can be supported only around its edge.

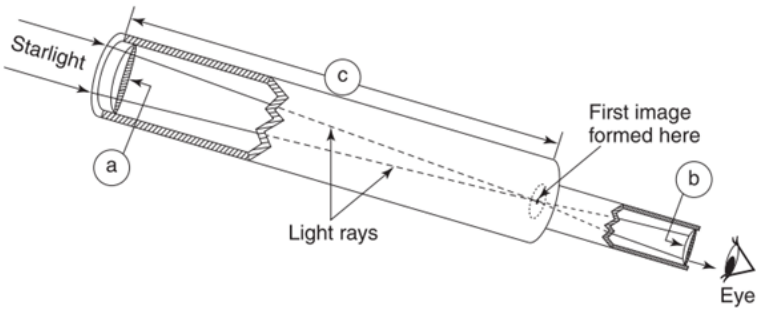


Figure 2.1: A refracting telescope. (a) The objective lens forms an image. (b) The eyepiece magnifies that image. (c) The focal length of the objective.

In a **reflecting telescope** the objective is a curved mirror at the bottom of the tube. A mirror can be supported across its whole back, and it brings all colours to the same focus, so every large modern telescope is a reflector. The point where the primary mirror alone forms its image is the *prime focus*. Since that point lies in the incoming beam, a second mirror is normally used to send the light somewhere more convenient:

- In a **Newtonian** telescope a small flat mirror turns the beam through 90°, out through the side of the tube (Figure 2.2).
- In a **Cassegrain** telescope a small convex mirror sends the beam back down the tube and through a hole in the primary (Figure 2.3). The convex secondary also lengthens the effective focal length, so a long focal length fits in a short tube.

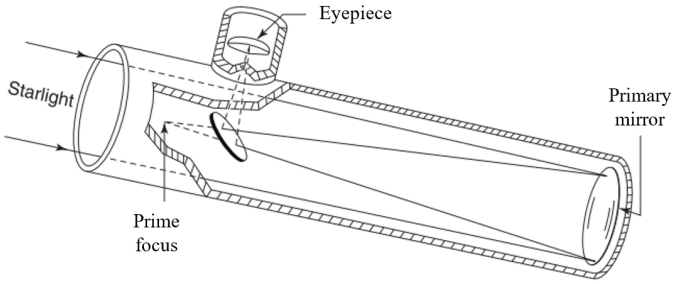


Figure 2.2: A Newtonian reflector.

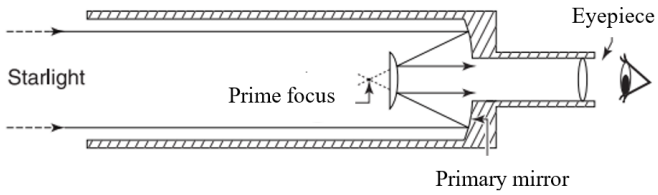


Figure 2.3: A Cassegrain reflector.

2.2 Light-gathering power

The light a telescope collects is proportional to the *area* of its objective, that is, to D^2. A fully dark-adapted eye has a pupil about 6 to 7 mm across. Compared with a 6 mm pupil, a 150 mm telescope collects

$$\left(\frac{150}{6}\right)^2 = 625$$

times as much light. A flux ratio of 625 is $2.5 \log_{10} 625 = 7.0$ magnitudes, so stars seven magnitudes fainter come into view.

In general, if the naked-eye limit is m_{eye} with pupil diameter D_{eye}, the faintest star visible through a telescope of aperture D is

$$m_{\text{lim}} = m_{\text{eye}} + 5\log_{10}\frac{D}{D_{\text{eye}}}$$

With $m_{\text{eye}} = 6$ and $D_{\text{eye}} = 6\,\text{mm}$ this is $m_{\text{lim}} \approx 2.1 + 5\log_{10} D\,[\text{mm}]$. The formula describes *visual* observing. A camera can add up light for minutes or hours and reaches far fainter objects with the same aperture.

Example 2.2.1 (Limiting magnitude)**.** From a dark site the naked-eye limit is 6.5 with a 6 mm pupil. What is the faintest star visible in a 200 mm telescope? In a city where the naked-eye limit is only 3.5, what does the same telescope reach?

Solution. Dark site: $m_{\text{lim}} = 6.5 + 5\log_{10}(200/6) = 6.5 + 7.6 = 14.1$. In the city the gain of the telescope is the same 7.6 magnitudes, so $m_{\text{lim}} = 3.5 + 7.6 = 11.1$. The telescope does not remove light pollution; it only adds its own gain to whatever the sky allows.

2.3 Focal length, focal ratio and image scale

Parallel rays from a distant star are brought to a point one **focal length** f behind the objective. The surface where images form is the *focal plane*. The **focal ratio** is

$$N = \frac{f}{D}, \tag{2.1}$$

written 'f/N'. A '150 mm $f/8$' telescope has $D = 150\,\text{mm}$ and $f = 8 \times 150 = 1200\,\text{mm}$.

Two stars separated by a small angle θ on the sky form images separated by a distance s on the focal plane (Figure 2.4), where

$$s = f\,\theta \qquad (\theta \text{ in radians}). \tag{2.2}$$

The size of the image depends only on the focal length, not on the aperture.

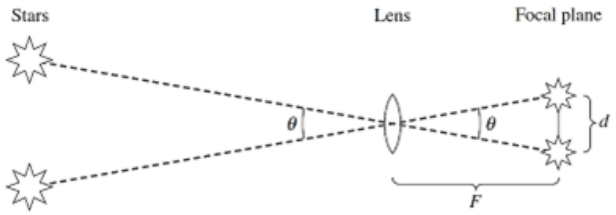


Figure 2.4: Two stars an angle θ apart are imaged a distance $s = f\theta$ apart on the focal plane.

The number of arcseconds that correspond to one millimetre on the focal plane is called the **plate scale**. Putting $s = 1\,\text{mm}$ in Equation (2.2) and converting radians to arcseconds,

$$\text{plate scale} = \frac{206\,265}{f\,[\text{mm}]}\ \text{arcsec per mm}$$

A *long* focal length gives a *small* plate scale and therefore a large, detailed image.

Example 2.3.1 (The Moon at the focus of the Yerkes refractor)**.** The Yerkes refractor has $D = 1.02\,\text{m}$ and focal ratio $f/19$. Find its plate scale and the diameter of the image of the full Moon (angular diameter $31'$). Compare with the human eye, whose focal length is about 17 mm.

Solution. $f = 19 \times 1.02 = 19.4\,\text{m} = 19\,400\,\text{mm}$, so the plate scale is $206\,265/19\,400 = 10.6''$ per mm. The Moon is $31 \times 60 = 1860''$ across, giving an image $1860/10.6 = 175\,\text{mm}$ wide, about the size of a dinner plate.

For the eye the scale is $206\,265/17 \approx 12\,100''$ per mm, and the Moon's image on the retina is only $1860/12\,100 = 0.15\,\text{mm}$ across.

2.4 Magnification and field of view

For visual use an **eyepiece** of focal length f_e is placed so that its focus coincides with that of the objective. Light from a star then leaves the eyepiece as a parallel beam again, but tilted at a larger angle than it entered (Figure 2.5). The **angular magnification** is

$$M = \frac{f_\text{obj}}{f_\text{e}}$$

Changing the eyepiece changes the magnification; nothing else about the telescope changes.

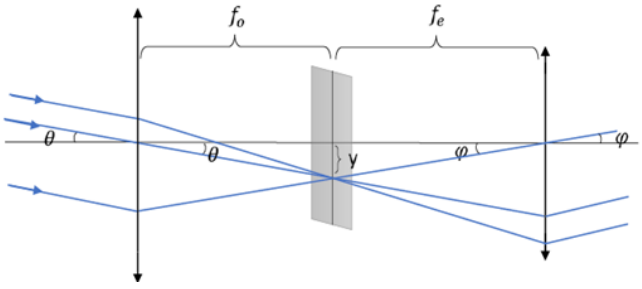


Figure 2.5: Angular magnification. Light entering at angle θ leaves the eyepiece at the larger angle ϕ.

Three related quantities are often asked for.

- **Field of view.** Every eyepiece has an *apparent field*, the angular width of the circle of light you see when you look into it (typically 40° to 70°).

The patch of sky actually visible, the *true field*, is

$$\text{true field} = \frac{\text{apparent field}}{M}. \tag{2.3}$$

More magnification always means a smaller piece of sky.

- **Exit pupil.** The beam leaving the eyepiece has diameter D/M. If this is larger than the pupil of the eye (about 7 mm), light is wasted, so the lowest useful magnification is about $D/7$ mm.
- **Highest useful magnification.** Beyond roughly $2D$ (with D in millimetres) the image only becomes larger and dimmer, with no new detail, because of the diffraction limit described below.



Figure 2.6: The field of view is the diameter of the circle of sky visible in the eyepiece.

Example 2.4.1 (Choosing an eyepiece)**.** A 150 mm $f/8$ reflector is used with a 25 mm eyepiece whose apparent field is 50°. Find the magnification, the true field and the exit pupil. Does the whole Moon fit in the field?

Solution. $f_{\text{obj}} = 8 \times 150 = 1200$ mm, so $M = 1200/25 = 48$. The true field is $50°/48 = 1.04° = 62'$, twice the diameter of the Moon, so the whole disc fits comfortably. The exit pupil is $150/48 = 3.1$ mm, smaller than the pupil of the eye, so no light is lost.

Measuring a field of view by drift timing

If the telescope is not driven, the rotation of the Earth carries stars across the field. The sky turns through 360° in one sidereal day, which is $15.04''$ per second of time on the celestial equator. A star at declination δ moves on a smaller circle and drifts more slowly, at $15.04'' \cos\delta$ per second. Timing a star as it crosses the centre of the field therefore measures the field diameter:

$$\text{field} = 15.04'' \times t\,[\text{s}] \times \cos\delta. \tag{2.4}$$

Example 2.4.2 (Will the Moon fit?)**.** A telescope has aperture 200 mm and focal length 2000 mm. With the drive switched off, a star at declination +20° takes 96 seconds to cross the field of view through its centre. Can the whole full Moon be seen at once?

Solution. From Equation (2.4),

$$\text{field} = 15.04'' \times 96 \times \cos 20^\circ = 1357'' = 22.6'.$$

The full Moon is about $30'$ across, so it does *not* fit. Notice that the aperture and focal length were not needed. Olympiad problems often include extra data, and part of the skill is recognising which numbers matter.

Example 2.4.3 (A crater on the Moon [adapted from IOAA 2007, short question 1.13])**.** What is the smallest aperture that can resolve a lunar crater 80 km across, in light of wavelength 500 nm? The Moon is 3.844×10^5 km away. Can the crater be seen with the naked eye?

Solution. The crater subtends $\theta = 80/3.844 \times 10^5 = 2.08 \times 10^{-4}$ rad (about $43''$). Setting this equal to the Rayleigh limit $1.22\lambda/D$,

$$D = \frac{1.22\lambda}{\theta} = \frac{1.22 \times 5.0 \times 10^{-7}}{2.08 \times 10^{-4}} = 2.9 \times 10^{-3}\,\text{m} = 2.9\,\text{mm}.$$

The pupil of the eye is larger than this, so in principle the crater is just resolvable without a telescope, although in practice the eye's real resolution of about $60''$ makes it marginal.

2.5 Resolving power

Light is a wave, and a wave passing through a circular opening spreads out slightly. The image of a star is therefore never a perfect point but a small disc surrounded by faint rings. Two stars can just be told apart when the centre of one disc falls on the first dark ring of the other (Figure 2.7). This **Rayleigh criterion** gives the smallest resolvable angle:

$$\theta_{\min} = 1.22\,\frac{\lambda}{D} \qquad \text{(radians)}$$

A larger aperture or a shorter wavelength gives sharper images. For visible light ($\lambda = 550$ nm) this is $\theta_{\min} \approx 138''/D$ [mm].

From the ground, turbulence in the atmosphere (*seeing*) blurs images to about $1''$ regardless of aperture, unless adaptive optics is used. This is one reason for putting telescopes in space.

Example 2.5.1 (Diffraction limits)**.** Find the diffraction limit at 550 nm of (a) the eye with a 6 mm pupil, (b) a 100 mm telescope, (c) the 2.4 m Hubble Space Telescope. Can the 100 mm telescope split a double star with a separation of $1.5''$?

Solution. Using $\theta_{\min} = 1.22\lambda/D$ and $1\,\text{rad} = 206\,265''$:

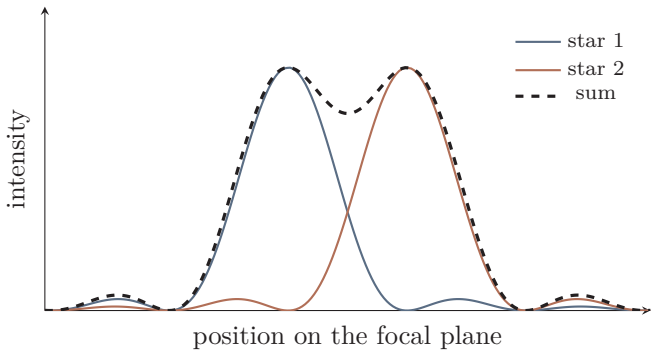


Figure 2.7: Two stars at the Rayleigh limit (schematic). The peak of each pattern lies on the first minimum of the other, and the combined profile shows a shallow dip between them.

(a) $1.22 \times 5.5 \times 10^{-7} / 0.006 = 1.12 \times 10^{-4}$ rad $= 23''$. (The real eye manages about $60''$ because of the spacing of cells in the retina.)

(b) $1.22 \times 5.5 \times 10^{-7} / 0.100 = 6.7 \times 10^{-6}$ rad $= 1.4''$.

(c) $1.22 \times 5.5 \times 10^{-7} / 2.4 = 2.8 \times 10^{-7}$ rad $= 0.058''$.

The double star at $1.5''$ is just above the $1.4''$ limit of the 100 mm telescope, so it can be split, but only on a night of steady air.

2.6 Aberrations

Real optics do not form perfect images even apart from diffraction. The main faults are:

- **Chromatic aberration.** A simple lens bends blue light more than red, so the colours focus at different distances and bright objects show coloured fringes. Mirrors do not suffer from it. Refractors reduce it with objectives made of two or three lenses of different glass.
- **Spherical aberration.** A mirror with a spherical surface focuses rays from its edge closer than rays from its centre. A paraboloidal mirror cures this for objects on the axis.
- **Coma.** Away from the centre of the field a paraboloid turns star images into small comet-shaped smears. It is worst in telescopes with a small focal ratio.

2.7 Detectors: the CCD

Professional astronomy and most amateur imaging now use electronic detectors. A CCD or CMOS sensor is a grid of light-sensitive *pixels*, each a few micrometres

across. A photon that lands on a pixel releases an electron, and the electrons collected during the exposure are counted. Compared with the eye, such a detector records most of the photons that reach it and can keep collecting for as long as the shutter is open.

Each pixel of width w covers an angle on the sky called the **pixel scale**. It follows directly from the plate scale:

$$\text{pixel scale} = 206\,265'' \times \frac{w}{f}. \tag{2.5}$$

The pixel scale should be about half of the smallest detail the telescope and atmosphere deliver, so that the detail is properly sampled.

Because photons arrive at random, a measurement of N photons has an uncertainty of about $\sqrt{N}$. The signal-to-noise ratio of a bright source is therefore $N/\sqrt{N} = \sqrt{N}$, and it improves only as the square root of the exposure time: four times the exposure gives twice the signal-to-noise.

Example 2.7.1 (Imaging Jupiter)**.** A camera with $5.0\,\mu$m pixels is placed at the focus of a telescope with $f = 1200\,$mm and $D = 150\,$mm. (a) What is the pixel scale? (b) How many pixels span Jupiter when its angular diameter is $45''$? (c) Is the image well sampled?

Solution.

(a) Pixel scale $= 206\,265 \times (5.0 \times 10^{-3}\,\text{mm})/1200\,\text{mm} = 0.86''$ per pixel.

(b) $45/0.86 = 52$ pixels across the disc.

(c) The diffraction limit is $138/150 = 0.92''$, about one pixel. Good sampling needs about two pixels per resolution element, so this image is *undersampled*. Doubling the focal length with an extra lens would fix it.

2.8 Radio telescopes and interferometers

The Rayleigh formula applies at every wavelength. Radio waves are hundreds of thousands of times longer than light waves, so a single radio dish has poor resolution even when it is very large. The remedy is to combine the signals from two or more dishes separated by a **baseline** B. Such an **interferometer** has the resolution of a single telescope as wide as the baseline,

$$\theta \approx \frac{\lambda}{B}, \tag{2.6}$$

although it collects only as much light as the dishes it actually has.

Example 2.8.1 (Seeing the black hole at the Galactic centre [IOAA 2010, short problem 8])**.** The black hole at the centre of the Milky Way has a mass of $4\times10^6\,M_\odot$ and is 8.5 kpc away. The radius of the event horizon of a black hole is about 3 km for each solar mass. An interferometer has dishes on opposite sides of the Earth, giving a baseline equal to the Earth's diameter. At what wavelength must it observe in order to resolve the event horizon?

Solution. The diameter of the horizon is $2 \times 3 \times 4 \times 10^6 = 2.4 \times 10^7$ km, and the distance is $8500 \times 3.086 \times 10^{13} = 2.62 \times 10^{17}$ km, so the horizon subtends

$$\theta = \frac{2.4 \times 10^7}{2.62 \times 10^{17}} = 9.1 \times 10^{-11}\,\text{rad} \approx 19 \text{ micro-arcseconds}.$$

With aperture $D = 2R_\oplus = 1.27 \times 10^7$ m, the Rayleigh criterion gives

$$\lambda = \frac{\theta D}{1.22} = \frac{(9.1 \times 10^{-11})(1.27 \times 10^7)}{1.22} = 9.6 \times 10^{-4}\,\text{m} \approx 1\,\text{mm}.$$

The observation must be made at millimetre wavelengths or shorter. This is exactly what the Event Horizon Telescope does, at 1.3 mm. For comparison, a single 100 m dish observing the 21 cm line resolves only $1.22 \times 0.21/100 = 2.6 \times 10^{-3}$ rad $= 9'$, no better than the naked eye.

2.9 Summary

Light gathering	$\propto D^2$
Limiting magnitude (visual)	$m_{\text{lim}} = m_{\text{eye}} + 5\log_{10}(D/D_{\text{eye}})$
Focal ratio	$N = f/D$
Image size	$s = f\theta$
Plate scale	$206\,265/f$ [mm] arcsec per mm
Magnification	$M = f_{\text{obj}}/f_{\text{e}}$
True field	apparent field$/M$
Exit pupil	D/M
Drift timing	field $= 15.04'' \times t\,[\text{s}] \times \cos\delta$
Resolution	$\theta_{\text{min}} = 1.22\,\lambda/D$
Pixel scale	$206\,265'' \times w/f$
Interferometer	$\theta \approx \lambda/B$

Table 2.1: Formulas of Chapter 2.

CHAPTER 3

Stars, Galaxies and the Expanding Universe

Chapter 1 showed how to measure the distance, luminosity and temperature of a star. This chapter uses those measurements to answer bigger questions: how stars are born, how they live and die, how they move, how they gather into galaxies, and how we know that the universe is expanding.

3.1 Spectra and the Doppler effect

Spread the light of a star into its colours and the smooth blackbody curve is found to be crossed by dark **absorption lines**. Each chemical element absorbs and emits light only at its own set of wavelengths, so the lines reveal what the outer layers of the star are made of. A hot, thin gas cloud shows the same lines bright against a dark background (**emission lines**). Three lines appear again and again in problems: hydrogen Hα at 656.3 nm, the Lyman-α line of hydrogen at 121.6 nm, and the radio line of neutral hydrogen at 21 cm.

Which lines are strongest depends mainly on temperature, and this is the basis of the **spectral classes** O, B, A, F, G, K, M, running from the hottest stars (O, above 30 000 K) to the coolest (M, about 3000 K). The Sun is a G star.

The Doppler shift

When a source moves along the line of sight, every line in its spectrum is displaced. If the rest wavelength is λ_0 and the observed wavelength is λ, then for speeds much smaller than light

$$\frac{\lambda - \lambda_0}{\lambda_0} = \frac{v_r}{c}$$

where v_r is the **radial velocity**. A source moving away has $v_r > 0$ and its lines move to longer wavelengths (*redshift*); an approaching source shows a *blueshift*. Only the component of the velocity along the line of sight matters.

When v_r is a sizeable fraction of c, the exact relation of special relativity must be used:

$$\frac{\lambda}{\lambda_0} = \sqrt{\frac{1 + v_r/c}{1 - v_r/c}} \qquad \Longleftrightarrow \qquad \frac{v_r}{c} = \frac{(\lambda/\lambda_0)^2 - 1}{(\lambda/\lambda_0)^2 + 1}. \tag{3.1}$$

Example 3.1.1 (Radial velocity from one line)**.** The Hα line of a star is observed at 656.52 nm. Is the star approaching or receding, and how fast?

Solution. $\Delta\lambda = 656.52 - 656.28 = +0.24\,\text{nm}$, a redshift, so the star is receding. Its speed is $v_r = c\,\Delta\lambda/\lambda_0 = (3.00 \times 10^5)(0.24/656.28) = 110\,\text{km s}^{-1}$.

Motion across the sky

The radial velocity is only half of a star's motion. The other half, the *tangential velocity* v_t, shows up as a slow drift in the star's position called **proper motion** μ, measured in arcseconds per year. For a star at distance d,

$$v_t\,[\text{km s}^{-1}] = 4.74\;\mu\,[''\,\text{yr}^{-1}]\;d\,[\text{pc}], \tag{3.2}$$

where the factor 4.74 converts AU per year into kilometres per second. The total speed through space is $v = \sqrt{v_r^2 + v_t^2}$.

3.2 The Hertzsprung–Russell diagram

Plot the luminosity of many stars against their surface temperature and they do not scatter at random. They fall into a few well-defined groups (Figure 3.1). This plot, the **Hertzsprung–Russell (HR) diagram**, is the single most useful picture in stellar astronomy. By a convention that is now too old to change, temperature increases to the *left*.

- About nine stars in ten lie on the **main sequence**, a band running from hot, luminous stars at the upper left to cool, faint ones at the lower right. These stars are fusing hydrogen into helium in their cores. The Sun is one of them.
- **Giants** and **supergiants** are cool but very luminous. By the Stefan–Boltzmann law a cool surface can be luminous only if it is enormous, so these stars have radii of tens to hundreds of solar radii.
- **White dwarfs** are hot but very faint, so they must be small: about the size of the Earth.

The main sequence is a sequence of mass

Where a star sits on the main sequence is decided by one number, its mass. More massive stars are hotter and far more luminous. Over the range from about 0.5 to 10 $M_\odot$,

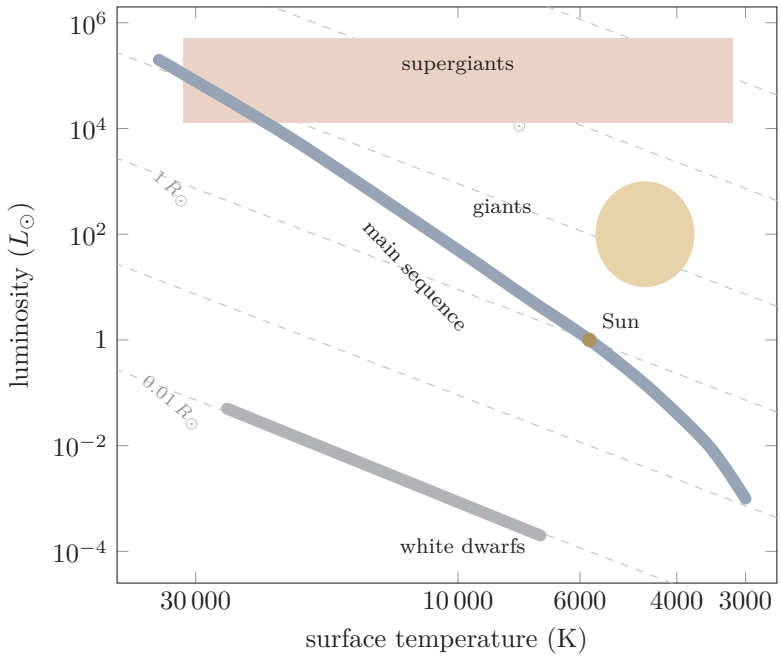


Figure 3.1: A schematic HR diagram. Dashed lines join stars of equal radius, from $L = 4\pi R^2 \sigma T^4$: stars in the upper right are huge, those in the lower left are tiny.

$$\frac{L}{L_\odot} \approx \left(\frac{M}{M_\odot}\right)^{3.5}$$

The fuel available is proportional to M, and it is used at a rate L, so the time a star can remain on the main sequence scales as M/L:

$$t_{\mathrm{MS}} \approx 10^{10}\,\mathrm{yr}\left(\frac{M}{M_\odot}\right)^{-2.5}. \tag{3.3}$$

Massive stars are the spendthrifts of the galaxy: they have more fuel but burn through it far faster.

Example 3.2.1 (How long can the Sun shine? [adapted from IOAA 2010, short problem 5])**.** Suppose the Sun could convert 0.8% of its whole mass into energy at its present luminosity. How long would it shine? Then explain why the true main-sequence lifetime is about ten times shorter, and use Equation (3.3) to find the lifetime of a $10\,M_\odot$ star.

Solution. The energy available would be $E = 0.008\,M_\odot c^2 = 0.008 \times (1.989 \times 10^{30})(2.998 \times 10^8)^2 = 1.43 \times 10^{45}$ J, which at the Sun's present luminosity lasts

$$t = \frac{E}{L_\odot} = \frac{1.43 \times 10^{45}}{3.828 \times 10^{26}} = 3.7 \times 10^{18}\,\mathrm{s} \approx 1.2 \times 10^{11}\,\mathrm{yr}.$$

This is an upper limit. Fusing hydrogen into helium releases about 0.7% of the mass, and only the innermost tenth of the Sun's hydrogen is hot enough to fuse, so the real lifetime is close to 10^{10} years.

For $10\,M_\odot$: $t_{\rm MS} = 10^{10} \times 10^{-2.5} = 3.2 \times 10^7$ yr, three hundred times shorter than the Sun's.

3.3 The birth of stars

Stars form inside cold, dense clouds of molecular hydrogen and dust (Figure 3.2). In such a cloud two effects compete: gravity pulls the gas inward, and the thermal motion of the molecules (pressure) pushes it outward. Gravity wins if the cloud is massive enough. The critical mass is called the **Jeans mass**,

$$M_J = \left(\frac{5kT}{G\mu m_{\rm H}}\right)^{3/2} \left(\frac{3}{4\pi\rho}\right)^{1/2}, \tag{3.4}$$

where T and ρ are the temperature and density of the gas, $m_{\rm H}$ is the mass of a hydrogen atom and μ is the mean mass of a gas particle in units of $m_{\rm H}$ ($\mu \approx 2$ for molecular hydrogen). Cold, dense clouds have a small Jeans mass and collapse most easily.

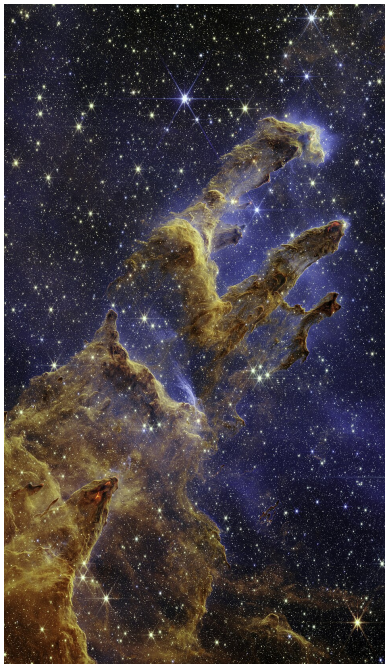


Figure 3.2: The Pillars of Creation in the Eagle Nebula, columns of cold gas and dust in which new stars are forming, imaged in the near infrared by the James Webb Space Telescope. Credit: NASA, ESA, CSA, STScI; J. DePasquale, A. Koekemoer, A. Pagan (STScI).

Once collapse begins, and while pressure is still unimportant, the cloud falls in on the **free-fall time**

$$t_{\mathrm{ff}} = \sqrt{\frac{3\pi}{32\,G\rho}}, \tag{3.5}$$

which depends only on the density. As the collapsing fragment (a *protostar*) shrinks, half of the gravitational energy released heats the gas and the other half is radiated away. A star powered only by this slow contraction could shine for the **Kelvin–Helmholtz time**

$$t_{\mathrm{KH}} \approx \frac{GM^2}{RL}, \tag{3.6}$$

about 30 million years for the Sun. Geology shows that the Earth is far older than that, which was the first clear sign that stars need another energy source. When the core of the protostar reaches roughly ten million kelvin, hydrogen fusion starts, the contraction stops, and the star settles on the main sequence in **hydrostatic equilibrium**: at every depth the pressure pushing outward balances the weight of the layers above.

Example 3.3.1 (Will this cloud collapse?)**.** A dense core in a molecular cloud has $T = 10\,\mathrm{K}$ and contains 10^4 hydrogen molecules per cubic centimetre. Find its Jeans mass and free-fall time.

Solution. The density is $\rho = (10^{10}\,\mathrm{m^{-3}})(2 \times 1.674 \times 10^{-27}\,\mathrm{kg}) = 3.35 \times 10^{-17}\,\mathrm{kg\,m^{-3}}$. Then

$$\frac{5kT}{G\mu m_{\mathrm{H}}} = \frac{5(1.381 \times 10^{-23})(10)}{(6.674 \times 10^{-11})(2)(1.674 \times 10^{-27})} = 3.09 \times 10^{15}\,\mathrm{kg\,m^{-1}},$$

$$M_J = (3.09 \times 10^{15})^{3/2} \left(\frac{3}{4\pi \times 3.35 \times 10^{-17}}\right)^{1/2} = 1.45 \times 10^{31}\,\mathrm{kg},$$

or about $7\,M_\odot$. A core with more than about seven solar masses will collapse. The free-fall time is

$$t_{\mathrm{ff}} = \sqrt{\frac{3\pi}{32(6.674 \times 10^{-11})(3.35 \times 10^{-17})}} = 1.15 \times 10^{13}\,\mathrm{s} \approx 3.6 \times 10^{5}\,\mathrm{yr},$$

which is very short compared with the life of a star.

3.4 The lives and deaths of stars

When the hydrogen in the core runs out, the core contracts and heats while the outer layers swell and cool: the star leaves the main sequence and becomes a red giant. What happens next depends on its mass (Figure 3.3).

Stars below about $8\,M_\odot$. The core becomes hot enough to fuse helium into carbon and oxygen, but no further. The outer layers drift away as a glowing shell called a *planetary nebula*, and the bare core is left as a **white dwarf**:

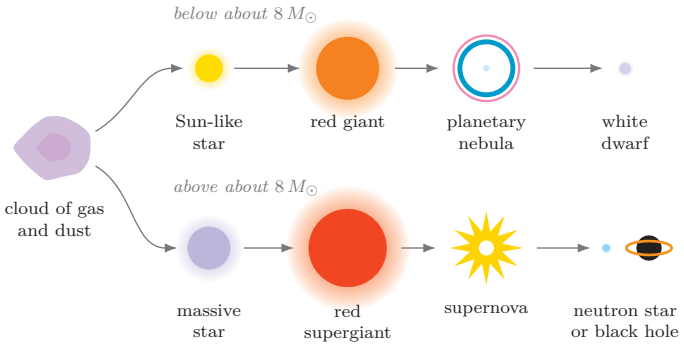


Figure 3.3: The life cycles of stars. Stars below about $8\,M_\odot$ end as white dwarfs; more massive stars explode as supernovae and leave a neutron star or a black hole.

roughly the mass of the Sun packed into the volume of the Earth, with a density near $10^9\,\mathrm{kg\,m^{-3}}$. A white dwarf has no energy source and simply cools. It is held up by *electron degeneracy pressure*, a quantum effect that does not depend on temperature. This support has a limit: no white dwarf can exceed the **Chandrasekhar mass**,

$$M_{\mathrm{Ch}} \approx 1.4\,M_\odot .$$

Stars above about $8\,M_\odot$. These fuse heavier and heavier elements until the core is iron. Fusing iron releases no energy, so the core collapses in less than a second. The collapse and rebound blow the rest of the star apart in a **core-collapse supernova** (Types II, Ib and Ic). The remnant is a **neutron star**, about 1.4 to $2\,M_\odot$ in a sphere only 10 to 15 km in radius, or, if the collapsing core is heavier than about $3\,M_\odot$, a **black hole**.

Type Ia supernovae. A white dwarf in a close binary can gain mass from its companion. If it approaches the Chandrasekhar mass, runaway carbon fusion destroys it completely. Because these explosions all start from nearly the same mass, they reach nearly the same peak luminosity ($M \approx -19.3$), and that makes them excellent *standard candles*. Do not confuse the two kinds: a Type Ia supernova is the explosion of a white dwarf; a core-collapse supernova is the death of a massive star, and its peak brightness varies too much to be a standard candle.

Black holes and gravitational redshift

The speed needed to escape from the surface of a body of mass M and radius R is $\sqrt{2GM/R}$. Setting this equal to the speed of light gives the **Schwarzschild radius**,

$$R_s = \frac{2GM}{c^2} \approx 2.95\,\mathrm{km} \times \frac{M}{M_\odot}$$

An object squeezed inside its own Schwarzschild radius is a black hole (Figure 3.4). Light climbing out of any gravitational field loses energy and is shifted to the red; for weak fields this **gravitational redshift** is

$$z_{\text{grav}} \approx \frac{GM}{Rc^2}. \tag{3.7}$$

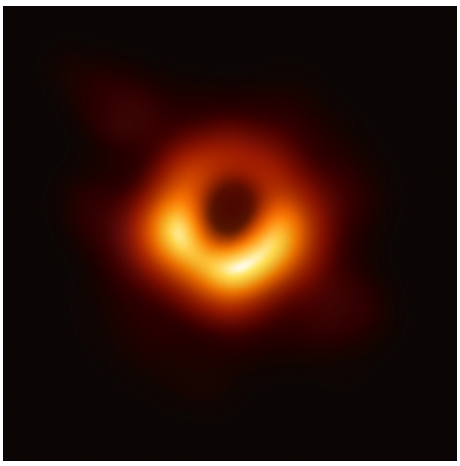


Figure 3.4: The first image of a black hole: the supermassive black hole of about $6.5 \times 10^9\, M_\odot$ at the centre of the galaxy M87, observed by the Event Horizon Telescope in 2017. The bright ring is hot gas whose light is bent around the dark central shadow. Credit: Event Horizon Telescope Collaboration (CC BY 4.0).

Example 3.4.1 (The Sun as a black hole [IOAA 2007, short question 1.14; IOAA 2010, short problem 2])**.** To what radius would the Sun have to be compressed for the escape speed from its surface to equal the speed of light? What would happen to the Earth's orbit if this occurred?

Solution.

$$R_s = \frac{2GM_\odot}{c^2} = \frac{2(6.674 \times 10^{-11})(1.989 \times 10^{30})}{(2.998 \times 10^8)^2} = 2.95 \times 10^3\,\text{m} = 2.95\,\text{km}.$$

The Earth's orbit would not change at all: outside a spherical body the gravitational field depends only on its mass, not on its size. The Earth would simply become dark and cold.

Example 3.4.2 (Gravitational redshift of a white dwarf)**.** A white dwarf has mass $0.6\, M_\odot$ and radius 6400 km. Find the gravitational redshift of its spectral lines, and the speed that would produce the same shift by the Doppler effect.

Solution.

$$z_{\rm grav} = \frac{GM}{Rc^2} = \frac{(6.674\times10^{-11})(0.6\times1.989\times10^{30})}{(6.4\times10^{6})(2.998\times10^{8})^2} = 1.4\times10^{-4},$$

equivalent to a Doppler speed $cz = 42\,\mathrm{km\,s^{-1}}$. The effect is easily measured, and it is one of the ways the masses of white dwarfs are found. For the Sun the same formula gives only 2.1×10^{-6}, or $0.6\,\mathrm{km\,s^{-1}}$.

3.5 Binary stars

At least half of all stars belong to binary systems, in which two stars orbit their common centre of mass. Binaries matter because they are the only direct way to measure stellar masses. Two results do almost all the work.

Kepler's third law for two bodies. If a is the semi-major axis of the relative orbit (the mean separation of the stars) and P the period,

$$P^2 = \frac{4\pi^2}{G(M_1+M_2)}\,a^3. \tag{3.8}$$

In units of years, astronomical units and solar masses this is simply

$$M_1 + M_2 = \frac{a^3}{P^2} \qquad (M_\odot,\ \mathrm{AU},\ \mathrm{yr})$$

The centre of mass. The stars lie on opposite sides of the centre of mass at distances a_1 and a_2, with $a = a_1 + a_2$ and

$$M_1 a_1 = M_2 a_2, \qquad \text{so also} \qquad \frac{M_1}{M_2} = \frac{a_2}{a_1} = \frac{v_2}{v_1}. \tag{3.9}$$

The heavier star moves on the smaller orbit and more slowly. The third law gives the *sum* of the masses; the centre-of-mass relation gives their *ratio*; together they give each mass.

Binaries are classified by how they are detected. In a *visual* binary both stars are seen separately. In a *spectroscopic* binary the stars are too close to separate, but their spectral lines shift back and forth by the Doppler effect as they orbit. In an *eclipsing* binary the orbit is seen edge-on and the stars pass in front of each other, producing regular dips in brightness.

Example 3.5.1 (A visual binary [IOAA 2010, short problem 4])**.** A binary star is 10 pc away and its orbit is seen face-on. The angular separation of the two stars varies between $1.0''$ and $7.0''$, and the period is 100 years. The semi-major axis of the orbit of one star about the centre of mass subtends $3.0''$. Find the mass of each star.

Solution. The smallest and largest separations are $a(1-e)$ and $a(1+e)$, so the semi-major axis of the relative orbit is the mean of the two, $4.0''$. At 10 pc one arcsecond corresponds to 10 AU, so $a = 40$ AU. (The eccentricity is $e = (7-1)/(7+1) = 0.75$.) The third law gives the total mass:

$$M_1 + M_2 = \frac{a^3}{P^2} = \frac{40^3}{100^2} = 6.4\,M_\odot.$$

One star has $a_1 = 3.0''$, so the other has $a_2 = 4.0'' - 3.0'' = 1.0''$. From $M_1 a_1 = M_2 a_2$ the masses are in the ratio $M_1 : M_2 = 1 : 3$, so

$$M_1 = 1.6\,M_\odot, \qquad M_2 = 4.8\,M_\odot.$$

The star with the larger orbit is the lighter one.

Example 3.5.2 (Sirius and its companion)**.** Sirius is a visual binary 2.64 pc away. The semi-major axis of the relative orbit subtends $7.5''$ and the period is 50.1 years. Find the total mass of the system.

Solution. $a = 7.5 \times 2.64 = 19.8$ AU, and

$$M_1 + M_2 = \frac{19.8^3}{50.1^2} = 3.1\,M_\odot.$$

The bright star Sirius A accounts for about two solar masses; the remaining solar mass belongs to its faint companion, the white dwarf Sirius B.

3.6 The Milky Way and other galaxies

The Sun is one of a few hundred billion stars in the **Milky Way**, a flattened, rotating disc about 30 kpc across with a central bulge, surrounded by a sparse spherical halo of old stars and globular clusters. The Sun lies in the disc, about 8 kpc from the centre, and orbits at about $220\,\mathrm{km\,s^{-1}}$. At the very centre is a black hole of four million solar masses, Sagittarius A*.

Galaxies come in three broad types (Figure 3.5):

- **Spirals**: rotating discs with spiral arms, rich in gas and dust, still forming stars. Barred spirals have a central bar. The Milky Way and the Andromeda galaxy are spirals.
- **Ellipticals**: smooth, rounded systems of old, red stars with very little gas. The largest galaxies known are giant ellipticals at the centres of clusters.
- **Irregulars**: small galaxies without a regular shape, such as the Magellanic Clouds.

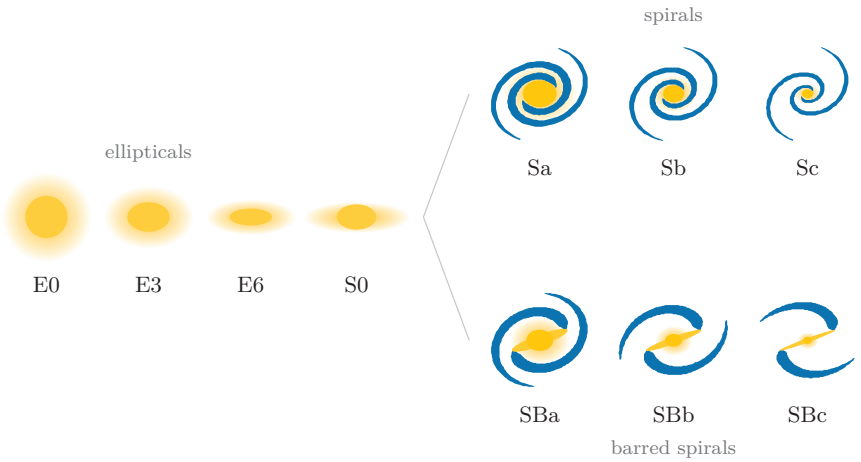


Figure 3.5: Hubble's tuning-fork classification of galaxies. Ellipticals are numbered by how flattened they look (E0 round, up to E7); S0 galaxies have a disc but no arms. Spirals (S) and barred spirals (SB) run from a large bulge and tightly wound arms (a) to a small bulge and open arms (c). Irregular galaxies fall outside the scheme. The diagram is a classification, not an evolutionary sequence.

Rotation curves and dark matter

A star on a circular orbit of radius r in a galaxy is held by the gravity of all the mass $M(r)$ inside its orbit. Setting the gravitational force equal to the centripetal force, $GM(r)m/r^2 = mv^2/r$, gives

$$M(r) = \frac{v^2\, r}{G}$$

A plot of orbital speed against radius is called a **rotation curve**. If most of a galaxy's mass were where most of its light is, near the centre, then far out $M(r)$ would be constant and the speed should fall as $v \propto r^{-1/2}$, just as it does for the planets of the solar system. That is not what is observed. Measured rotation curves stay flat as far out as they can be followed (Figure 3.6). A constant v means $M(r) \propto r$: the mass keeps growing where there is almost no light. This unseen material is called **dark matter**, and it outweighs the visible matter of a galaxy several times over.

Example 3.6.1 (Weighing the Milky Way)**.** Use the Sun's orbit ($r = 8.0\,\text{kpc}$, $v = 220\,\text{km}\,\text{s}^{-1}$) to estimate the mass of the Milky Way inside the Sun's orbit, and the time the Sun takes to complete one orbit.

Solution. $r = 8000 \times 3.086 \times 10^{16} = 2.47 \times 10^{20}\,\text{m}$. Then

$$M = \frac{v^2 r}{G} = \frac{(2.20 \times 10^5)^2 (2.47 \times 10^{20})}{6.674 \times 10^{-11}} = 1.8 \times 10^{41}\,\text{kg} \approx 9 \times 10^{10}\, M_\odot.$$

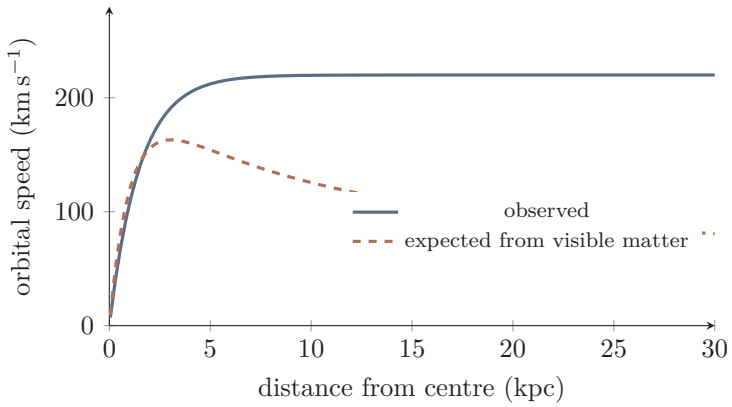


Figure 3.6: A schematic rotation curve of a spiral galaxy. The gap between the two curves is the evidence for dark matter.

The orbital period is $2\pi r/v = 7.1 \times 10^{15}\,\mathrm{s} \approx 220$ million years. The Sun has been round the galaxy about twenty times since it formed.

3.7 The expanding universe

In the 1920s it was discovered that the spectra of almost all galaxies are redshifted, and that the redshift grows with distance. The **redshift** is defined as

$$z = \frac{\lambda - \lambda_0}{\lambda_0}. \tag{3.10}$$

For nearby galaxies ($z \lesssim 0.1$) we may write $v \approx cz$, and the observations are summarised by **Hubble's law**:

$$v = H_0\, d$$

where $H_0 \approx 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ is the Hubble constant. Different methods currently give values between about 67 and 73; use the value given in the problem.

Hubble's law does not mean that we are at the centre of an explosion. Space itself is stretching, and every observer in every galaxy sees the others receding in the same way. The redshift of a distant galaxy is really a stretching of the light waves as they travel through expanding space, and for large z neither $v = cz$ nor the Doppler formula Equation (3.1) gives a true distance. For the redshifts met in CAAO problems, $v = cz$ and $d = cz/H_0$ are the intended tools unless the question says otherwise.

If the expansion had always gone on at today's rate, every galaxy would have been on top of every other a time $1/H_0$ ago. This **Hubble time** is an estimate of the age of the universe:

$$t_H = \frac{1}{H_0} = \frac{3.086 \times 10^{19}\,\mathrm{km}}{70\,\mathrm{km\,s^{-1}}} = 4.4 \times 10^{17}\,\mathrm{s} \approx 14 \text{ billion years}.$$

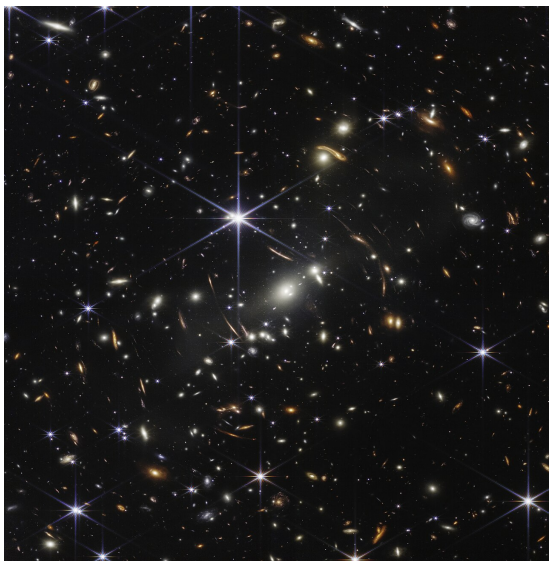


Figure 3.7: Webb's First Deep Field: the galaxy cluster SMACS 0723 and many more distant galaxies behind it, imaged by the James Webb Space Telescope in 2022. Almost every object in the picture is a galaxy. Credit: NASA, ESA, CSA, STScI.

Example 3.7.1 (Redshift, distance and size of a galaxy)**.** The Hα line ($\lambda_0 = 6563$ Å) of a galaxy is observed at 6727 Å, and the galaxy has an angular diameter of $1.5'$. Taking $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, find its redshift, recession speed, distance and physical diameter, and compare it with the Milky Way.

Solution. The redshift is $z = (6727 - 6563)/6563 = 0.0250$; it is positive, so the galaxy is receding, at $v \approx cz = 0.0250 \times 3.00 \times 10^5 = 7500\,\mathrm{km\,s^{-1}}$. Hubble's law gives

$$d = \frac{v}{H_0} = \frac{7500}{70} = 107\,\mathrm{Mpc}.$$

The angular diameter is $90''$, so by the small-angle formula

$$D = d\,\theta = 107\,\mathrm{Mpc} \times \frac{90}{206\,265} = 0.047\,\mathrm{Mpc} = 47\,\mathrm{kpc},$$

about one and a half times the diameter of the Milky Way.

Example 3.7.2 (From magnitude to mass)**.** A spiral galaxy 20 Mpc away has apparent magnitude $m = 12.0$. Its mass-to-light ratio is 4 in solar units. Estimate the mass of its stars. Ignore extinction.

Solution. Absolute magnitude: $M = m - 5\log_{10} d + 5 = 12.0 - 5\log_{10}(2.0 \times 10^7) + 5 = -19.5$.

Luminosity: $L/L_\odot = 10^{0.4(4.83+19.5)} = 5.4 \times 10^9$.

Mass: a mass-to-light ratio of 4 means $4\,M_\odot$ for every $L_\odot$, so the mass is $4 \times 5.4 \times 10^9 = 2.2 \times 10^{10}\,M_\odot$.

3.8 The cosmic distance ladder

No single method measures every distance in the universe. Each method works over a limited range and is calibrated using the one below it, like the rungs of a ladder (Figure 3.8).

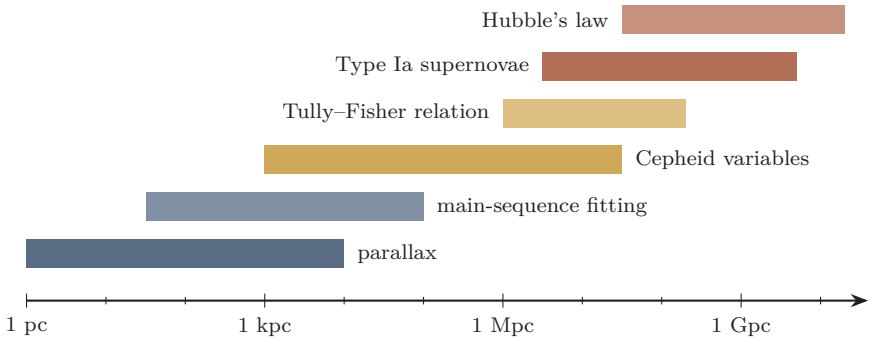


Figure 3.8: The distance ladder. Each bar shows roughly the range of distances over which a method is used; the overlaps allow each method to calibrate the next.

- **Parallax** is the only direct, geometric rung and the foundation of all the others.
- **Main-sequence fitting.** All the stars of a cluster are at the same distance. Sliding the cluster's main sequence vertically on the HR diagram until it matches a calibrated one gives the distance modulus $m - M$ directly.
- **Cepheid variables** are pulsating supergiants whose period of pulsation is tightly related to their luminosity: the longer the period, the more luminous the star. Measure the period, read off M, compare with m.
- **The Tully–Fisher relation.** The luminosity of a spiral galaxy rises with its rotation speed roughly as $L \propto v_{\max}^4$. The rotation speed can be measured from the width of the galaxy's 21 cm line.
- **Type Ia supernovae** have a known peak luminosity and are bright enough to be seen across billions of parsecs.
- **Hubble's law** turns a redshift into a distance once H_0 has been fixed by the rungs below.

An object of known luminosity, such as a Cepheid or a Type Ia supernova, is called a **standard candle**. Every standard-candle method ends with the same step: the distance modulus.

Example 3.8.1 (A Cepheid in another galaxy)**.** A Cepheid with a period of 10 days is found in a nearby galaxy, with mean apparent magnitude $m_V = 24.0$. The period–luminosity relation is $M_V = -2.43\,(\log_{10} P - 1) - 4.05$ with P in days. Find the distance of the galaxy, ignoring extinction.

Solution. For $P = 10$ days, $\log_{10} P = 1$ and $M_V = -4.05$. The distance modulus is $m - M = 28.05$, so

$$d = 10^{(28.05+5)/5} = 10^{6.61}\,\mathrm{pc} = 4.1\,\mathrm{Mpc}.$$

3.9 Summary

Doppler shift	$\Delta\lambda/\lambda_0 = v_r/c$
Tangential velocity	$v_t = 4.74\,\mu\,d \quad (\mathrm{km\,s^{-1}}, \mathrm{arcsec\,yr^{-1}}, \mathrm{pc})$
Mass–luminosity	$L/L_\odot \approx (M/M_\odot)^{3.5}$
Main-sequence lifetime	$t_{\mathrm{MS}} \approx 10^{10}\,\mathrm{yr}\,(M/M_\odot)^{-2.5}$
Jeans mass	$M_J = (5kT/G\mu m_{\mathrm{H}})^{3/2}(3/4\pi\rho)^{1/2}$
Free-fall time	$t_{\mathrm{ff}} = \sqrt{3\pi/32G\rho}$
Schwarzschild radius	$R_s = 2GM/c^2$
Binary stars	$M_1 + M_2 = a^3/P^2; \quad M_1 a_1 = M_2 a_2$
Enclosed mass	$M(r) = v^2 r/G$
Hubble's law	$v = H_0 d, \quad v \approx cz$
Hubble time	$t_H = 1/H_0$

Table 3.1: Formulas of Chapter 3.

CHAPTER 4

Spherical Astronomy

Look up on a clear night and the sky seems to be a great dome with the stars fixed to its inside. The dome is an illusion, since the stars are at very different distances, but it is a useful one. For finding where an object is in the sky, when it rises and sets, and how the sky changes with place and time, only directions matter. We may therefore imagine every star projected onto a sphere of arbitrary radius centred on the observer: the **celestial sphere**. Working on it needs the geometry of a sphere, which is where this chapter begins.

4.1 Geometry on a sphere

A plane that passes through the centre of a sphere cuts its surface in a **great circle**; any other plane cuts it in a **small circle** (Figure 4.1). The equator of the Earth is a great circle; every other parallel of latitude is a small circle. The shortest path between two points on a sphere is the arc of the great circle through them. Since only directions matter, lengths of arcs on the celestial sphere are measured as angles at the centre.

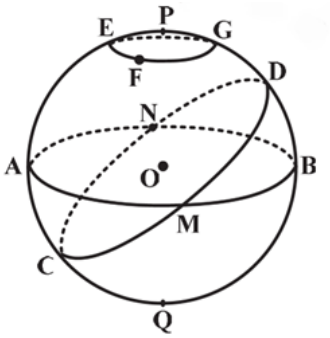


Figure 4.1: Great and small circles. $ANBM$, $CNDM$ and $APBQ$ are great circles; EFG is a small circle.

A **spherical triangle** is the figure formed by three arcs of great circles (Figure 4.2). By convention its angles are called A, B, C and the sides opposite them a, b, c. All six are angles. Unlike a plane triangle, the three angles of a spherical triangle always add up to *more* than 180°, and the excess measures

its area:

$$\text{Area} = (A + B + C - \pi)\, R^2 \qquad \text{(angles in radians)}, \tag{4.1}$$

where R is the radius of the sphere. On a sphere of unit radius the area is in steradians.

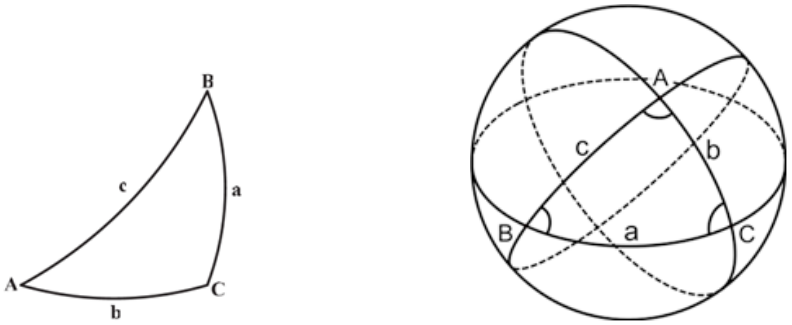


Figure 4.2: A spherical triangle with sides a, b, c and opposite angles A, B, C.

The four formulas

Four relations between the sides and angles solve almost every problem in spherical astronomy.

Sine formula

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \tag{4.2}$$

Cosine formula (three sides and one angle)

$$\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A \\ \cos b &= \cos a \cos c + \sin a \sin c \cos B \\ \cos c &= \cos a \cos b + \sin a \sin b \cos C \end{aligned} \tag{4.3}$$

Polar formula (three angles and one side)

$$\begin{aligned} -\cos A &= \cos B \cos C - \sin B \sin C \cos a \\ -\cos B &= \cos A \cos C - \sin A \sin C \cos b \\ -\cos C &= \cos A \cos B - \sin A \sin B \cos c \end{aligned} \tag{4.4}$$

Four-parts formula (four consecutive parts: side, angle, side, angle)

$$\cos a \cos C = \sin a \cot b - \sin C \cot B \tag{4.5}$$

Here a and C are the two *inner* parts of the four consecutive parts b, C, a, B.

Which formula to use depends on what is known. The cosine formula is the workhorse: it gives the third side from two sides and the angle between them, and it gives any angle from three sides. A word of caution about the sine formula: $\sin x = \sin(180^\circ - x)$, so it cannot tell an angle from its supplement. Whenever possible, find an unknown from its cosine.

Example 4.1.1 (Solving a spherical triangle)**.** In a spherical triangle $a = 34°46'$, $b = 27°22'$ and the included angle is $C = 72°31'$. Find c, A and B, and the area of the triangle on a sphere of unit radius.

Solution. Third side, from the cosine formula:

$$\cos c = (0.8215)(0.8881) + (0.5703)(0.4597)(0.3004) = 0.8083,$$

so $c = 36°04'$. The angles follow from the cosine formula rearranged:

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} = 0.3830 \qquad \Rightarrow \quad A = 67°29',$$
$$\cos B = \frac{\cos b - \cos a \cos c}{\sin a \sin c} = 0.6674 \qquad \Rightarrow \quad B = 48°08'.$$

The angles add up to $188°08'$, so the spherical excess is $8°08' = 0.142$ rad and the area is 0.142 steradians. As a check, the sum of the angles exceeds 180°, as it must.

Angular distance between two points

A direct use of the cosine formula gives the angle θ between two points on a sphere whose coordinates are known. For two stars with right ascensions α_1, α_2 and declinations δ_1, δ_2 (defined below),

$$\cos\theta = \sin\delta_1 \sin\delta_2 + \cos\delta_1 \cos\delta_2 \cos(\alpha_1 - \alpha_2)$$

The same formula, with latitude in place of declination and longitude in place of right ascension, gives the great-circle distance between two cities.

Example 4.1.2 (Sirius and Procyon)**.** Sirius is at $\alpha = 6^{\mathrm{h}}45.1^{\mathrm{m}}$, $\delta = -16°43'$ and Procyon at $\alpha = 7^{\mathrm{h}}39.3^{\mathrm{m}}$, $\delta = +5°13'$. How far apart are they on the sky?

Solution. The difference in right ascension is $54.2^{\mathrm{m}} = 13.55°$ (one minute of time is 0.25°). Then

$$\begin{aligned}\cos\theta &= \sin(-16.72°)\sin(5.22°) + \cos(-16.72°)\cos(5.22°)\cos(13.55°)\\ &= -0.0262 + 0.9272 = 0.9011,\end{aligned}$$

so $\theta = 25.7°$, a little more than the span of an outstretched hand at arm's length.

4.2 Position on the Earth

The Earth supplies the model for every coordinate system on a sphere. Its rotation axis meets the surface at the north and south poles, and the great circle midway between them is the **equator**.

- **Latitude** ϕ is the angular distance north or south of the equator, from $-90°$ to $+90°$ (Figure 4.3).
- **Longitude** l is measured along the equator, east or west from the meridian through Greenwich.

Because the Earth turns through 360° in 24 hours, longitude and many other angles in astronomy are often expressed in units of time: $1^{\mathrm{h}} = 15°$, $1^{\mathrm{m}} = 15'$, $1^{\mathrm{s}} = 15''$.

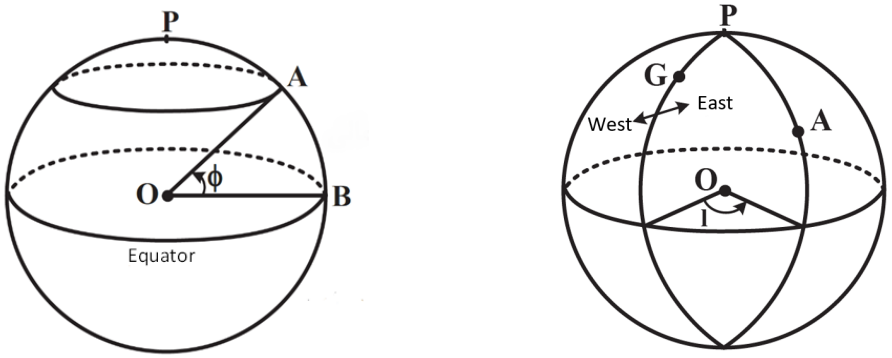


Figure 4.3: Latitude ϕ (left) and longitude l (right).

4.3 The horizontal system

The most natural coordinates are those of the observer's own surroundings (Figure 4.4). The point directly overhead is the **zenith** Z, the point directly below is the *nadir*, and the great circle halfway between them is the **horizon**.

- **Altitude** h is the angle of the object above the horizon, from 0° at the horizon to 90° at the zenith. Its complement $z = 90° - h$ is the **zenith distance**.
- **Azimuth** A is measured around the horizon. In this book, as in the IOAA syllabus, azimuth runs from north through east: north is 0°, east 90°, south 180° and west 270°.

Altitude and azimuth tell you exactly where to point a telescope, but they have two drawbacks: they are different for every observer, and they change continuously as the Earth rotates. A catalogue of the sky needs coordinates fixed to the stars.

4.4 The equatorial system

Extend the Earth's rotation axis until it meets the celestial sphere. The two points where it does so are the **north and south celestial poles** (NCP and SCP), and the whole sky appears to turn about them once a day. Project the Earth's equator outwards and it becomes the **celestial equator** (Figure 4.5).

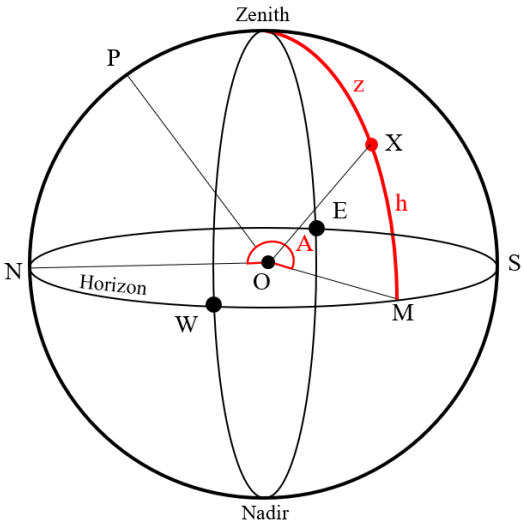


Figure 4.4: The horizontal (alt-azimuth) system. X is the star, h its altitude and z its zenith distance.

- **Declination** δ is the angular distance of an object north (+) or south (−) of the celestial equator, the exact counterpart of latitude.
- **Right ascension** α (RA) is the counterpart of longitude. It is measured eastward along the celestial equator from the **vernal equinox** γ, the point where the Sun crosses the equator going north in March, and it is given in hours, from 0^{h} to 24^{h}.

The right ascension and declination of a star do not depend on where the observer is or on the time of night. (They change slowly over decades because of precession, which is why catalogues quote an epoch such as J2000.)

The pole and the latitude

One simple fact ties the two systems together (Figure 4.6):

> The altitude of the celestial pole equals the latitude of the observer.

At the north pole of the Earth the NCP is overhead; at the equator it lies on the horizon. An observer in Toronto ($\phi = 43.7°$) sees it $43.7°$ above the north point, very close to the star Polaris.

The meridian and the hour angle

The great circle through the north point, the celestial pole, the zenith and the south point is the observer's **meridian**. As the sky turns, each star crosses the meridian twice a day. Crossing it at its highest point is **upper culmination** (or *transit*); crossing it at its lowest point, twelve hours later, is **lower culmination**, which for many stars happens below the horizon.

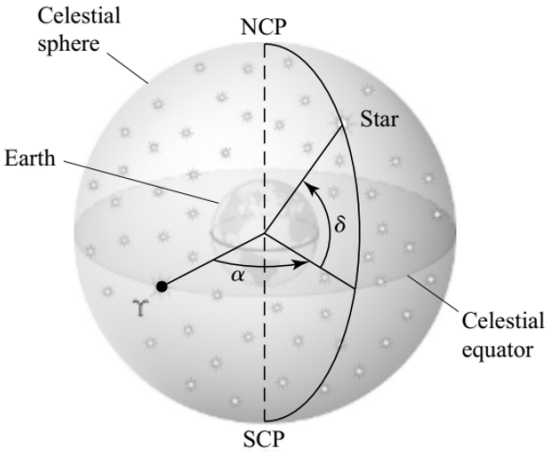


Figure 4.5: The equatorial system: right ascension α and declination δ.

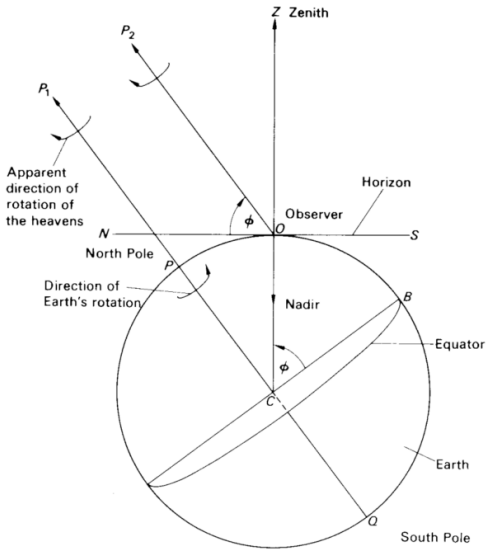


Figure 4.6: The altitude of the north celestial pole equals the observer's latitude ϕ.

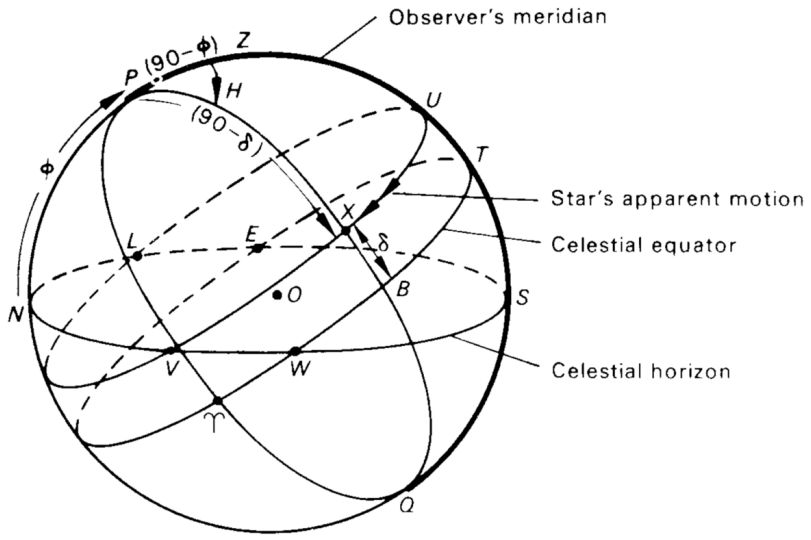


Figure 4.7: The horizontal and equatorial systems drawn together for one observer. The star X follows the parallel of declination UXV, rising at L, transiting at U and setting at V.

The **hour angle** H of a star is the angle at the pole between the meridian and the star, measured westward. It is the time elapsed since the star's upper culmination: $H = 0$ on the meridian, positive in the west and negative (or between 12^{h} and 24^{h}) in the east. Unlike the right ascension, the hour angle of a star increases steadily, by 24^{h} in one sidereal day.

Culmination

On the meridian everything lies in one plane and no spherical trigonometry is needed. For an observer in the northern hemisphere:

$$h_{\mathrm{upper}} = 90^\circ - |\phi - \delta|, \qquad h_{\mathrm{lower}} = \phi + \delta - 90^\circ. \tag{4.6}$$

If $\delta < \phi$ the star culminates south of the zenith, if $\delta > \phi$ north of it. At upper culmination the altitude is greatest and the zenith distance *smallest*; at lower culmination the zenith distance is greatest.

Circumpolar stars

A star whose lower culmination is still above the horizon never sets; it is **circumpolar**. From Equation (4.6) with $h_{\mathrm{lower}} > 0$, for a northern observer

$$\delta > 90^\circ - \phi \quad \text{(never sets)}, \qquad \delta < -(90^\circ - \phi) \quad \text{(never rises)}. \tag{4.7}$$

Figure 4.8 shows these two caps. At the poles of the Earth every visible star is circumpolar; at the equator none is.

Example 4.4.1 (A circumpolar star)**.** A circumpolar star has altitude 62° at upper culmination and 22° at lower culmination. Find its declination and the latitude of the observer, who is in the northern hemisphere.

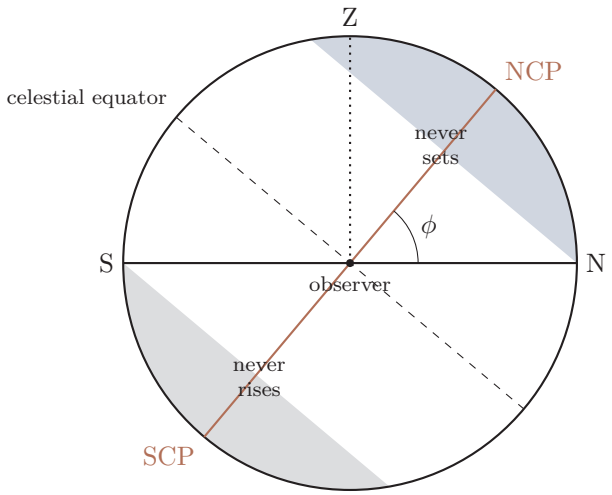


Figure 4.8: The celestial sphere seen from the west for an observer at latitude $\phi = 50°$. Stars within ϕ of the north celestial pole never set; stars within ϕ of the south celestial pole never rise.

Solution. From Equation (4.6), $\phi + \delta - 90° = 22°$, so $\phi + \delta = 112°$, and $90° - |\phi - \delta| = 62°$, so $|\phi - \delta| = 28°$. There are two solutions:

$$\phi = 70°,\ \delta = 42° \qquad \text{or} \qquad \phi = 42°,\ \delta = 70°.$$

In the first the star culminates south of the zenith, in the second north of it. A single pair of altitudes cannot distinguish them; the observer also needs to note on which side of the zenith the upper culmination occurs. The second solution is the one people forget.

4.5 The astronomical triangle

Away from the meridian, the link between the two systems is the spherical triangle whose corners are the pole P, the zenith Z and the star X (Figure 4.9). Its parts are worth learning by heart:

side PZ	$90° - \phi$ (co-latitude)
side PX	$90° - \delta$ (polar distance)
side ZX	$90° - h = z$ (zenith distance)
angle at P	hour angle H
angle at Z	azimuth A (star in the east) or $360° - A$ (star in the west)

Applying the cosine formula to the side ZX, and then to the side PX, gives the two most used equations in spherical astronomy:

$$\sin h = \sin\phi \sin\delta + \cos\phi \cos\delta \cos H$$

$$\sin\delta = \sin\phi \sin h + \cos\phi \cos h \cos A$$

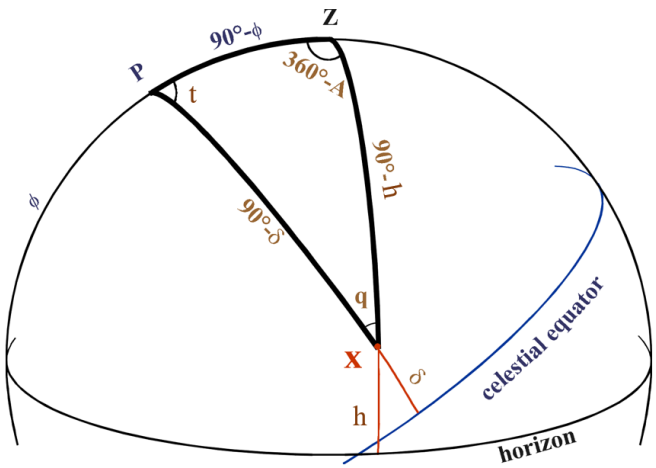


Figure 4.9: The astronomical triangle PZX.

The first gives the altitude of a star at any hour angle. The second gives its azimuth once the altitude is known. Because $\cos A = \cos(360^\circ - A)$, the second equation cannot tell east from west; decide from the hour angle (east of the meridian before transit, west after).

Rising and setting

At rising and setting the altitude is zero. Putting $h = 0$ in the two equations gives

$$\cos H_0 = -\tan\phi \tan\delta \qquad\qquad \cos A_0 = \frac{\sin\delta}{\cos\phi}$$

The star rises at hour angle $-H_0$ and sets at $+H_0$, so it is above the horizon for a time $2H_0$. It rises at azimuth A_0 and sets at $360^\circ - A_0$. A star on the celestial equator ($\delta = 0$) rises due east, sets due west and is up for exactly twelve hours, at every latitude. If $|\tan\phi\tan\delta| > 1$ there is no solution: the star is circumpolar or never rises. These formulas ignore atmospheric refraction, which lifts objects near the horizon by about half a degree.

4.6 Sidereal time

Because the hour angle of every star grows at the same steady rate, the sky itself can be used as a clock. The **local sidereal time** (LST) is defined as the hour angle of the vernal equinox. Since right ascension is measured from the vernal equinox in the opposite direction to hour angle, for any star

$$\mathrm{LST} = H + \alpha$$

In particular a star is on the meridian ($H = 0$) when the sidereal time equals its right ascension. This is the most practical statement in the chapter: *the right ascension on your meridian is your sidereal time.*

Two further facts are needed.

- **Sidereal and solar days.** The Earth turns once relative to the stars in a *sidereal day* of $23^{\mathrm{h}}56^{\mathrm{m}}04^{\mathrm{s}}$. Because the Earth also moves along its orbit, it must turn a little further to bring the Sun back to the meridian, and the *solar day* is 24 hours. Stars therefore rise about four minutes earlier each night, which adds up to two hours a month and a full day in a year.
- **Longitude.** Observers further east see every star cross the meridian earlier. If l is the longitude measured positive eastward, $\mathrm{LST} = \mathrm{GST} + l$, where GST is the sidereal time at Greenwich.

A useful anchor: around 22 September the Sun is at $\alpha = 12^{\mathrm{h}}$, so at local midnight the sidereal time is about 0^{h}. It then gains about two hours per month.

Example 4.6.1 (Solar and sidereal days [adapted from IOAA 2007, short question 1.3])**.** How many sidereal days are there in 183 solar days?

Solution. In one year the Earth turns once more relative to the stars than relative to the Sun, so a year of 365.24 solar days contains 366.24 sidereal days. Therefore

$$183 \times \frac{366.24}{365.24} = 183.50 \text{ sidereal days.}$$

In half a year the stars gain half a turn on the Sun: the constellations on the meridian at midnight are those that were there at noon six months earlier.

Example 4.6.2 (Rising and setting of a star)**.** Ignoring refraction, find the sidereal times and azimuths of rising and setting of Arcturus ($\alpha = 14^{\mathrm{h}}15.7^{\mathrm{m}}$, $\delta = +19°11'$) for an observer at latitude 45° N.

Solution. Hour angle at the horizon:

$$\cos H_0 = -\tan(45°)\tan(19.18°) = -0.3479 \;\Rightarrow\; H_0 = 110.36° = 7^{\mathrm{h}}21.4^{\mathrm{m}}.$$

The star rises at $H = -7^{\mathrm{h}}21.4^{\mathrm{m}}$ and sets at $H = +7^{\mathrm{h}}21.4^{\mathrm{m}}$. Using $\mathrm{LST} = H + \alpha$,

$$\mathrm{LST}_{\mathrm{rise}} = 14^{\mathrm{h}}15.7^{\mathrm{m}} - 7^{\mathrm{h}}21.4^{\mathrm{m}} = 6^{\mathrm{h}}54.3^{\mathrm{m}},$$
$$\mathrm{LST}_{\mathrm{set}} = 14^{\mathrm{h}}15.7^{\mathrm{m}} + 7^{\mathrm{h}}21.4^{\mathrm{m}} = 21^{\mathrm{h}}37.1^{\mathrm{m}}.$$

Azimuth:

$$\cos A_0 = \frac{\sin 19.18°}{\cos 45°} = 0.4646 \;\Rightarrow\; A_0 = 62.3°.$$

The star rises at azimuth 62.3° (north of east) and sets at $360° - 62.3° = 297.7°$. It is above the horizon for $2H_0 = 14^{\mathrm{h}}43^{\mathrm{m}}$ of each sidereal day.

Example 4.6.3 (What is on the meridian?)**.** Estimate the local sidereal time at 21:00 local time on 21 December. Where is Orion ($\alpha \approx 5.5^{\mathrm{h}}$) at that moment?

Solution. 21 December is 90 days after 22 September, so at local midnight the sidereal time is about $90 \times (24^{\mathrm{h}}/365) = 5.9^{\mathrm{h}}$. Three hours earlier, at 21:00, it is about 2.9^{h}. Orion's hour angle is $H = \mathrm{LST} - \alpha = 2.9 - 5.5 = -2.6^{\mathrm{h}}$: it is in the south-east, and will cross the meridian in about two and a half hours.

4.7 The Sun, the ecliptic and the seasons

Seen from the Earth, the Sun moves once around the celestial sphere each year along a great circle called the **ecliptic**, which is simply the plane of the Earth's orbit (Figure 4.10). The ecliptic is tilted to the celestial equator by the **obliquity** $\varepsilon = 23.44°$, because the Earth's rotation axis is tilted by that angle. The two circles cross at the equinoxes.

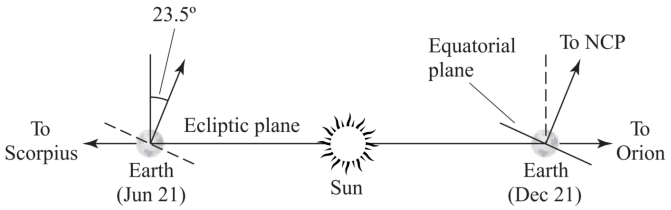


Figure 4.10: The plane of the Earth's orbit seen edge-on.

Event	Date	$\alpha_\odot$	$\delta_\odot$
March (vernal) equinox	about 20 March	0^{h}	$0°$
June solstice	about 21 June	6^{h}	$+23.44°$
September equinox	about 22 September	12^{h}	$0°$
December solstice	about 21 December	18^{h}	$-23.44°$

The Sun's changing declination is the cause of the seasons: when $\delta_\odot$ is positive the Sun is higher at noon and above the horizon for longer in the northern hemisphere. Between the tabulated dates a good approximation is

$$\delta_\odot \approx 23.44° \sin\left[\frac{360°}{365}\,(N - 81)\right], \tag{4.8}$$

where N is the day of the year. Everything derived above for stars applies to the Sun with $\delta = \delta_\odot$: its noon altitude is $90° - |\phi - \delta_\odot|$, and the length of the day is $2H_0$.

Example 4.7.1 (The longest and shortest days in Toronto)**.** For Toronto ($\phi = 43.65°$ N) find the noon altitude of the Sun and the length of the day at the two solstices, ignoring refraction and the size of the Sun's disc.

Solution. June solstice, $\delta_\odot = +23.44°$: noon altitude $= 90° - 43.65° + 23.44° = 69.8°$. For the day length, $\cos H_0 = -\tan 43.65° \tan 23.44° = -0.4137$, so $H_0 = 114.4°$ and the day lasts $2 \times 114.4/15 = 15.3$ hours.

December solstice, $\delta_\odot = -23.44°$: noon altitude $= 90° - 43.65° - 23.44° = 22.9°$; $\cos H_0 = +0.4137$, $H_0 = 65.6°$, and the day lasts 8.7 hours. The two day lengths add up to 24 hours, as symmetry requires.

Example 4.7.2 (Sunrise on the shortest day [based on IOAA 2007, short question 1.1])**.** An observer is at latitude 42.5° N and longitude 71° W, in a time zone whose standard meridian is 75° W. At what zone time does the Sun rise on the December solstice ($\delta_\odot = -23.5°$)? Ignore refraction, the size of the Sun's disc and the equation of time.

Solution. $\cos H_0 = -\tan(42.5°)\tan(-23.5°) = +0.3984$, so $H_0 = 66.5° = 4^{\mathrm{h}}26^{\mathrm{m}}$. The Sun rises $4^{\mathrm{h}}26^{\mathrm{m}}$ before local noon, at 7:34 local solar time.

The observer is 4° east of the standard meridian, and each degree is 4 minutes of time, so local time is 16 minutes *ahead* of zone time. The Sun therefore rises at $7{:}34 - 0{:}16 = 7{:}18$ zone time. The day lasts only $2H_0 = 8^{\mathrm{h}}52^{\mathrm{m}}$.

Example 4.7.3 (The midnight Sun)**.** What is the lowest northern latitude from which the Sun can be seen at midnight, ignoring refraction? On what date?

Solution. The Sun is visible at midnight if its lower culmination is above the horizon, that is if it is circumpolar: $\delta_\odot > 90° - \phi$. The Sun's declination is greatest at the June solstice, +23.44°, so the condition is first met at

$$\phi = 90° - 23.44° = 66.56° \text{ N},$$

the Arctic Circle, on about 21 June. Farther north the midnight Sun lasts longer, and at the pole it lasts six months.

Solar time

Clocks follow the Sun, not the stars. **Apparent solar time** is the hour angle of the Sun plus 12 hours, so that the Sun crosses the meridian at 12:00. The real Sun is a slightly irregular clock, because the Earth's orbit is elliptical and the ecliptic is tilted to the equator. **Mean solar time** smooths out the irregularity. The difference,

$$\text{equation of time} = \text{apparent solar time} - \text{mean solar time}, \tag{4.9}$$

varies through the year between about −14 and +16 minutes. **Universal Time** (UT) is the mean solar time at Greenwich, and the local mean time at east longitude l is UT $+ l$. Civil clocks keep the mean time of a standard meridian for a whole *time zone* (for Toronto, 75° W, which is UT $- 5^{\mathrm{h}}$), with an extra hour added in summer.

4.8 The ecliptic system

For the Sun, Moon and planets, which stay close to the ecliptic, it is convenient to use the ecliptic itself as the reference circle (Figure 4.11).

- **Ecliptic latitude** β is the angular distance north or south of the ecliptic.
- **Ecliptic longitude** λ is measured eastward along the ecliptic from the vernal equinox, from $0°$ to $360°$.

The Sun always has $\beta = 0$, and its longitude increases by about one degree per day.

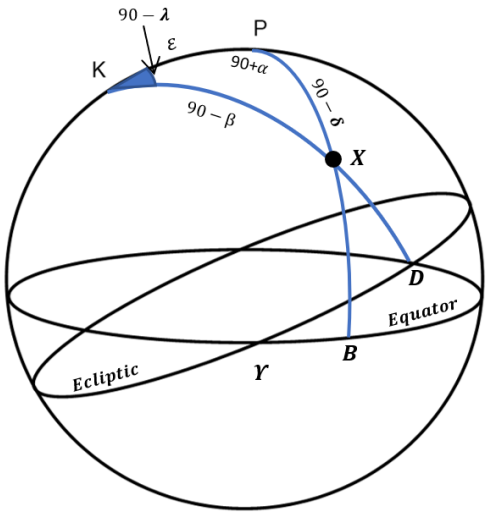


Figure 4.11: The triangle formed by the celestial pole P, the pole of the ecliptic K and the star X, used to convert between equatorial and ecliptic coordinates.

The conversion uses the spherical triangle formed by the celestial pole P, the pole of the ecliptic K and the star X. Its sides are $KP = \varepsilon$, $PX = 90° - \delta$ and $KX = 90° - \beta$; the angle at P is $90° + \alpha$ and the angle at K is $90° - \lambda$. The cosine formula applied to KX and then to PX gives

$$\sin\beta = \sin\delta\cos\varepsilon - \cos\delta\sin\varepsilon\sin\alpha$$

$$\sin\delta = \sin\beta\cos\varepsilon + \cos\beta\sin\varepsilon\sin\lambda$$

and the sine formula adds the useful relation $\cos\beta\cos\lambda = \cos\delta\cos\alpha$, which fixes the quadrant of λ. For the Sun ($\beta = 0$) the second equation reduces to $\sin\delta_\odot = \sin\varepsilon\sin\lambda_\odot$.

4.9 Summary

Cosine formula	$\cos a = \cos b \cos c + \sin b \sin c \cos A$
Angular separation	$\cos \theta = \sin \delta_1 \sin \delta_2 + \cos \delta_1 \cos \delta_2 \cos \Delta\alpha$
Pole	altitude of pole $= \phi$
Culmination	$h_{\text{upper}} = 90^\circ - \lvert\phi - \delta\rvert, \quad h_{\text{lower}} = \phi + \delta - 90^\circ$
Circumpolar	$\delta > 90^\circ - \phi$
Altitude	$\sin h = \sin \phi \sin \delta + \cos \phi \cos \delta \cos H$
Azimuth	$\sin \delta = \sin \phi \sin h + \cos \phi \cos h \cos A$
Rising and setting	$\cos H_0 = -\tan \phi \tan \delta, \quad \cos A_0 = \sin \delta / \cos \phi$
Sidereal time	$\text{LST} = H + \alpha$
Sun	$\sin \delta_\odot = \sin \varepsilon \sin \lambda_\odot, \quad \varepsilon = 23.44^\circ$

Table 4.1: Formulas of Chapter 4.

CHAPTER 5

The Solar System and Celestial Mechanics

This chapter is about motion under gravity. It starts with what we actually see, the wandering of the planets across the sky, and then turns to the laws that explain it: Kepler's three laws and Newton's law of gravitation. The same few equations describe planets, comets, moons, spacecraft and double stars.

5.1 Planets in our sky

All the planets orbit the Sun in the same direction and nearly in the same plane, so from the Earth they are always found close to the ecliptic. How a planet behaves in our sky depends on whether its orbit is inside or outside ours.

- **Inferior planets** (Mercury and Venus) orbit inside the Earth's orbit. They never stray far from the Sun and are seen only in the evening sky after sunset or the morning sky before sunrise.
- **Superior planets** (Mars and beyond) orbit outside the Earth's orbit and can appear at any angle from the Sun, including directly opposite it.

The angle between a planet and the Sun, as seen from the Earth, is the planet's **elongation**. Particular values of the elongation have names (Figure 5.1):

Configuration	Elongation	Applies to
Superior conjunction	0° (beyond the Sun)	all planets
Inferior conjunction	0° (in front of the Sun)	inferior
Greatest elongation	largest possible	inferior
Quadrature	90°	superior
Opposition	180°	superior

A superior planet at opposition is at its closest to the Earth, rises at sunset and is visible all night; this is the best time to observe it. An inferior planet is best seen near greatest elongation. At that moment the line of sight from the Earth is tangent to the planet's orbit, so the Earth–planet–Sun angle is a right angle (Figure 5.2). For circular orbits, with the planet's orbital radius a in AU,

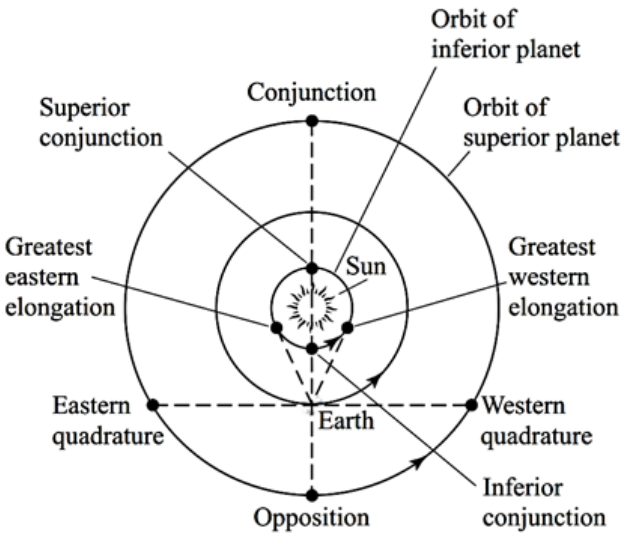


Figure 5.1: Planetary configurations.

$$\sin(\text{greatest elongation}) = a$$

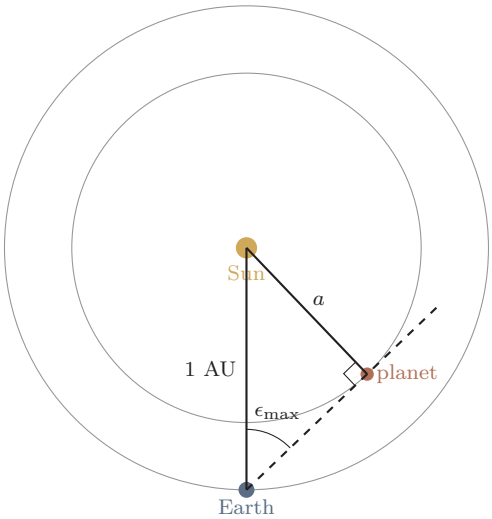


Figure 5.2: Greatest elongation of an inferior planet. The line of sight is tangent to the planet's orbit, so the triangle has a right angle at the planet and $\sin \epsilon_{\max} = a/1\,\text{AU}$.

Synodic period

The time a planet takes to go once round the Sun relative to the stars is its **sidereal period** P. What we observe from the moving Earth is different: the

time between two successive oppositions, or two successive inferior conjunctions, is the **synodic period** S. In one day the Earth moves through $360^\circ/E$ and the planet through $360^\circ/P$, where $E = 365.256$ days is the Earth's sidereal period. The faster body gains on the slower at the difference of these rates, and a synodic period is the time needed to gain one full lap:

$$\frac{1}{S} = \left| \frac{1}{P} - \frac{1}{E} \right|$$

The same formula gives the time between alignments of any two bodies orbiting a common centre, with E replaced by the period of the second body.

Example 5.1.1 (Venus [adapted from IOAA 2007, short question 1.2])**.** The greatest elongation of Venus is observed to be 46°, and its sidereal period is 224.70 days. Assuming circular orbits, find the radius of its orbit and its synodic period.

Solution. At greatest elongation the triangle Sun–Venus–Earth has its right angle at Venus, so $a = 1\,\mathrm{AU} \times \sin 46^\circ = 0.72\,\mathrm{AU}$.

Venus moves faster than the Earth, so

$$\frac{1}{S} = \frac{1}{224.70} - \frac{1}{365.256} = 1.7126 \times 10^{-3}\,\mathrm{d}^{-1} \quad \Rightarrow \quad S = 583.9 \text{ days}.$$

Venus returns to the same configuration only about every 19 months, and because it never moves more than 46° from the Sun it can never be seen in the middle of the night.

Example 5.1.2 (Great oppositions of Mars [IOAA 2010, short problem 9])**.** Mars was at a 'great' opposition, one that occurs when Mars is near its perihelion, on 28 August 2003. Estimate the date of the next great opposition. The sidereal period of Mars is 686.98 days.

Solution. Oppositions are separated by the synodic period,

$$S = \left(\frac{1}{365.256} - \frac{1}{686.98} \right)^{-1} = 779.9 \text{ days} = 2.135 \text{ years}.$$

For an opposition to be a great one, Mars must be back near its perihelion, which means the Earth must be back near the same point of its own orbit: a whole number of synodic periods must be close to a whole number of years. Trying multiples of 2.135 years gives 2.14, 4.27, 6.41, 8.54, 10.68, 12.81, 14.95: the seventh is within 0.05 of a year of a whole number. So the next great opposition comes after $7S = 5459.5$ days.

From 28 August 2003 to 28 August 2018 is $15 \times 365 + 4 = 5479$ days (four leap days), so the opposition falls 19 days earlier, around 8 or 9 August 2018. (The true date was 27 July 2018; the difference arises because Mars moves much faster than average near perihelion, which our circular-orbit estimate ignores.)

From one configuration to the next

The relative motion of the Earth and a planet is uniform (for circular orbits), so the time between two configurations is the fraction of the synodic period corresponding to the change in the *Sun-centred* angle between the two bodies. Be careful: this is not the same as the change in elongation, which is an Earth-centred angle. At opposition the Sun-centred angle between the Earth and the planet is zero. At quadrature the triangle Sun–Earth–planet has its right angle at the Earth, so the angle at the Sun is θ with $\cos\theta = 1/a$.

Example 5.1.3 (From quadrature to opposition)**.** Jupiter ($a = 5.203\,\mathrm{AU}$, synodic period 398.9 days) is at western quadrature today. How long until its next opposition?

Solution. At quadrature the Sun-centred angle between the Earth and Jupiter is $\theta = \cos^{-1}(1/5.203) = 78.9^\circ$. The Earth must close this angle, which takes

$$t = \frac{78.9^\circ}{360^\circ} \times 398.9 = 87 \text{ days}.$$

It would be a mistake to say that the elongation changes by 90° and therefore the time is a quarter of the synodic period (100 days). The two answers agree only for a very distant planet, for which $\theta \to 90^\circ$.

Retrograde motion

Planets normally drift eastward against the stars. Around opposition, however, a superior planet appears to stop, move westward for some weeks, stop again, and resume its eastward motion. Nothing has happened to the planet: the Earth, on its faster inner orbit, is overtaking it, just as a slower car seems to move backwards when you pass it on a highway. Inferior planets show retrograde motion around inferior conjunction for the same reason.

5.2 Newton's law of gravitation

Every two masses attract each other with a force proportional to the product of the masses and inversely proportional to the square of the distance r between their centres:

$$F = \frac{G m_1 m_2}{r^2}, \qquad G = 6.674 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}. \tag{5.1}$$

Three consequences are used constantly.

- The acceleration of a small body at distance r from a mass M is $g = GM/r^2$, whatever the mass of the small body.
- The gravitational potential energy of the pair is

  $$U = -\frac{GMm}{r}. \tag{5.2}$$

  It is zero when the bodies are infinitely far apart and becomes more negative as they approach.

- Gravity acts along the line joining the two bodies, so it exerts no torque about the central body. The **angular momentum** of an orbiting body about the centre is therefore constant. For a body of mass m at distance r whose velocity component perpendicular to the radius is $v_\perp$, the angular momentum is $L = mrv_\perp$.

Throughout this chapter the central body is assumed to be much more massive than the orbiting one, so that it can be treated as fixed. The case of two comparable masses was treated in Chapter 3.

5.3 Kepler's laws

Kepler found his three laws from observations of Mars, decades before Newton showed that all three follow from the law of gravitation.

First law: orbits are ellipses

Each planet moves on an ellipse with the Sun at one focus. An ellipse is described by its **semi-major axis** a (half of its longest diameter) and its **eccentricity** e, a number between 0 and 1 that measures how elongated it is (Figure 5.3). The distance from the centre of the ellipse to each focus is ae. A circle is an ellipse with $e = 0$.

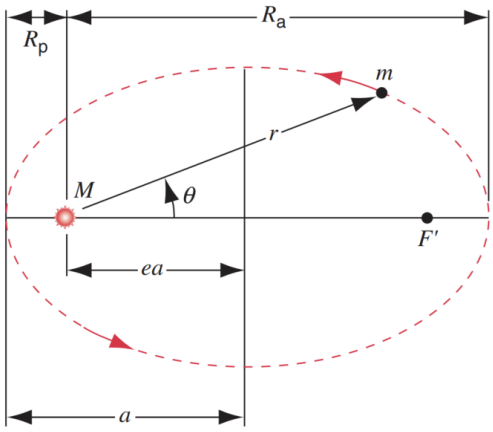


Figure 5.3: A planet of mass m on an elliptical orbit around the Sun, of mass M.

The point of the orbit nearest the Sun is the **perihelion** and the farthest is the **aphelion** (for orbits around the Earth, *perigee* and *apogee*). Their distances from the Sun are

$$r_p = a(1 - e) \qquad r_a = a(1 + e)$$

so that $r_p + r_a = 2a$ and $e = (r_a - r_p)/(r_a + r_p)$. At a general point, with the angle θ measured at the Sun from the direction of perihelion,

$$r = \frac{a(1-e^2)}{1 + e\cos\theta}. \tag{5.3}$$

Second law: equal areas in equal times

The line joining a planet to the Sun sweeps out equal areas in equal intervals of time (Figure 5.4). In a short time Δt the line sweeps a thin triangle of area $\frac{1}{2} r\,(v_\perp \Delta t)$, so the rate at which area is swept is

$$\frac{\Delta A}{\Delta t} = \frac{1}{2}\, r\, v_\perp = \frac{L}{2m}. \tag{5.4}$$

Since the angular momentum L is constant, so is this rate: the second law is the conservation of angular momentum. A planet therefore moves fastest at perihelion and slowest at aphelion. At those two points the velocity is exactly perpendicular to the radius, which gives a very useful relation:

$$r_p\, v_p = r_a\, v_a$$

Third law: the harmonic law

The square of the orbital period is proportional to the cube of the semi-major axis. For a circular orbit of radius r the proof takes two lines. Gravity supplies the centripetal force,

$$\frac{GMm}{r^2} = \frac{mv^2}{r}, \tag{5.5}$$

and the speed is one circumference per period, $v = 2\pi r/T$. Substituting,

$$T^2 = \frac{4\pi^2}{GM}\, r^3. \tag{5.6}$$

The same result holds for an ellipse with r replaced by the semi-major axis a:

$$T^2 = \frac{4\pi^2}{GM}\, a^3$$

The mass of the orbiting body does not appear. Measuring T and a for *any* satellite gives the mass of the body it orbits; this is how the masses of the Sun, the planets and galaxies are found. For bodies orbiting the Sun, with T in years and a in AU, the constant is one and the law is simply $T^2 = a^3$ (Table 5.1).

Example 5.3.1 (A comet [adapted from IOAA 2007, short questions 1.6 and 1.7])**.** A comet has perihelion distance 0.50 AU and aphelion distance 31.5 AU. Find its orbital period, the eccentricity of its orbit, and the area swept out each year by the line joining it to the Sun.

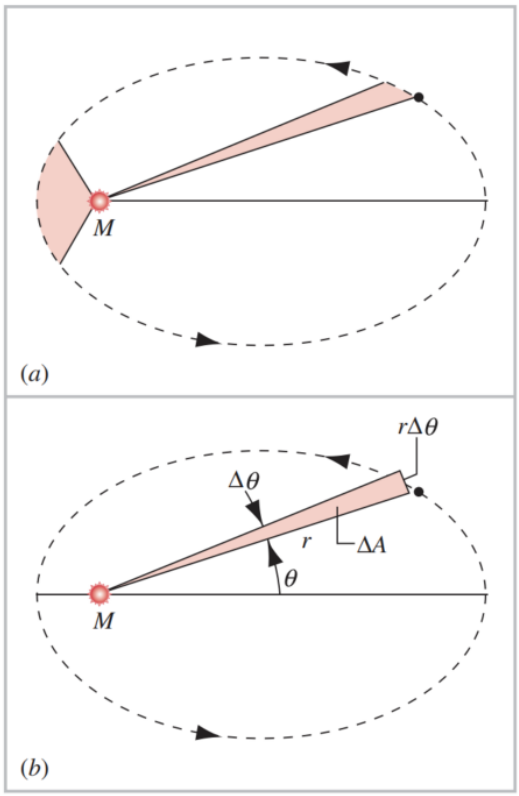


Figure 5.4: (a) The shaded areas are swept in equal times. (b) In a time t the line from the Sun to the planet turns through an angle θ and sweeps an area A.

Planet	a (AU)	T (yr)	T^2/a^3
Mercury	0.387	0.241	1.00
Venus	0.723	0.615	1.00
Earth	1.000	1.000	1.00
Mars	1.524	1.881	1.00
Jupiter	5.203	11.86	1.00
Saturn	9.537	29.46	1.00
Uranus	19.19	84.01	1.00
Neptune	30.07	164.8	1.00

Table 5.1: Kepler's third law for the planets.

Solution. The semi-major axis is $a = (0.50+31.5)/2 = 16.0\,\text{AU}$, so by the third law $T = 16.0^{3/2} = 64$ years. The eccentricity is $e = (31.5-0.50)/(31.5+0.50) = 0.969$.

By the second law the area is swept at a constant rate, so the rate is the whole area of the ellipse divided by the period. The semi-minor axis is $b = a\sqrt{1-e^2} = 16.0 \times 0.248 = 3.97\,\text{AU}$ (see Appendix A), the area is $\pi ab = \pi \times 16.0 \times 3.97 = 199.5\,\text{AU}^2$, and

$$\frac{\Delta A}{\Delta t} = \frac{199.5}{64} = 3.1\,\text{AU}^2 \text{ per year.}$$

5.4 Energy and the vis-viva equation

The total energy of an orbiting body, kinetic plus potential, is constant:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}. \tag{5.7}$$

For an elliptical orbit its value depends only on the semi-major axis:

$$E = -\frac{GMm}{2a}. \tag{5.8}$$

Equating the two expressions gives the speed at any point of the orbit, the **vis-viva equation**:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

This one equation contains several familiar results.

- **Circular orbit** ($r = a$): $v_\text{c} = \sqrt{GM/r}$.
- **Escape** ($a \to \infty$, total energy zero): $v_\text{esc} = \sqrt{2GM/r} = \sqrt{2}\,v_\text{c}$.
- **Perihelion and aphelion**: put $r = a(1 \mp e)$ to get

$$v_p = \sqrt{\frac{GM}{a}\,\frac{1+e}{1-e}}, \qquad v_a = \sqrt{\frac{GM}{a}\,\frac{1-e}{1+e}}. \tag{5.9}$$

For orbits around the Sun there is a shortcut that avoids G and M altogether. The Earth's orbital speed is $\sqrt{GM_\odot/1\,\text{AU}} = 29.8\,\text{km s}^{-1}$, so with r and a in AU

$$v = 29.8\,\text{km s}^{-1} \times \sqrt{\frac{2}{r} - \frac{1}{a}}. \tag{5.10}$$

The four kinds of orbit

The sign of the total energy decides the shape of the orbit (Figure 5.5).

Orbit	Eccentricity	Total energy	Speed at distance r
Circle	$e = 0$	$E < 0$	$v = v_{\mathrm{c}}$
Ellipse	$0 < e < 1$	$E < 0$	$v < v_{\mathrm{esc}}$
Parabola	$e = 1$	$E = 0$	$v = v_{\mathrm{esc}}$
Hyperbola	$e > 1$	$E > 0$	$v > v_{\mathrm{esc}}$

A body on a parabola or a hyperbola is not bound: it passes the Sun once and never returns. On a hyperbola it still has speed left over when it is far away. The first known interstellar visitor, 1I/'Oumuamua, was recognised as such because its orbit was strongly hyperbolic.

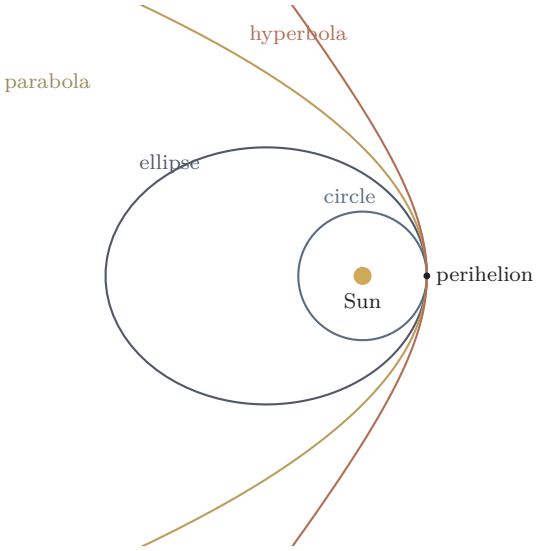


Figure 5.5: Four orbits with the same perihelion. The faster the body moves at perihelion, the more open the orbit.

Example 5.4.1 (Halley's comet)**.** Halley's comet has $a = 17.8\,\mathrm{AU}$ and $e = 0.967$. Find its period, its perihelion and aphelion distances, and its speed at each.

Solution. $T = 17.8^{3/2} = 75$ years. The extreme distances are $r_p = 17.8 \times 0.033 = 0.59\,\mathrm{AU}$, inside the orbit of Venus, and $r_a = 17.8 \times 1.967 = 35.0\,\mathrm{AU}$, beyond Neptune. From Equation (5.10),

$$v_p = 29.8\sqrt{\frac{2}{0.587} - \frac{1}{17.8}} = 54.5\,\mathrm{km\,s^{-1}},$$

$$v_a = 29.8\sqrt{\frac{2}{35.0} - \frac{1}{17.8}} = 0.91\,\mathrm{km\,s^{-1}}.$$

The comet moves sixty times faster at perihelion than at aphelion, in the ratio r_a/r_p as the second law requires, and so spends almost all of its time far from the Sun.

Example 5.4.2 (A walk around an asteroid [IOAA 2010, short problem 6])**.** A spherical asteroid has diameter 2.2 km and density $2.2\,\mathrm{g\,cm^{-3}}$. Could an astronaut walk all the way round its equator in 2.2 hours?

Solution. The radius is 1100 m and the density $2200\,\mathrm{kg\,m^{-3}}$, so the mass is $M = \frac{4}{3}\pi(1100)^3(2200) = 1.23 \times 10^{13}$ kg. The speed of a circular orbit just above the surface is

$$v_\mathrm{c} = \sqrt{\frac{GM}{R}} = \sqrt{\frac{(6.674 \times 10^{-11})(1.23 \times 10^{13})}{1100}} = 0.86\,\mathrm{m\,s^{-1}}.$$

To complete the circumference in 2.2 hours the astronaut must move at

$$v = \frac{2\pi \times 1100}{2.2 \times 3600} = 0.87\,\mathrm{m\,s^{-1}}.$$

This is slightly *more* than the orbital speed. At that speed the astronaut would no longer be pressed against the ground but would drift off into orbit, so the walk is impossible. On small bodies the difficulty is not moving fast enough but staying on the surface.

5.5 Satellites and transfer orbits

For a satellite of the Earth the same formulas apply with $GM_\oplus = 3.986 \times 10^{14}\,\mathrm{m^3\,s^{-2}}$ and distances measured from the *centre* of the Earth ($R_\oplus = 6371$ km). Two cases are worth knowing.

- A satellite in **low Earth orbit**, about 400 km up, moves at $\sqrt{GM_\oplus/r} = 7.7\,\mathrm{km\,s^{-1}}$ and completes an orbit in about 92 minutes. The escape speed from the surface is $11.2\,\mathrm{km\,s^{-1}}$.
- A **geostationary** satellite orbits above the equator with a period of one sidereal day (86 164 s), so it stays above the same point on the ground. From the third law its orbital radius is 42 160 km, which is 35 800 km above the surface.

The Hohmann transfer

The most economical way to move between two circular orbits of radii r_1 and r_2 is along half of an ellipse that touches both (Figure 5.6). The transfer ellipse has its perihelion on the inner orbit and its aphelion on the outer one, so

$$a_\mathrm{t} = \frac{r_1 + r_2}{2}, \tag{5.11}$$

and the journey lasts half the period of that ellipse. Two short engine burns are needed: one to leave the first orbit and one to match speed with the second.

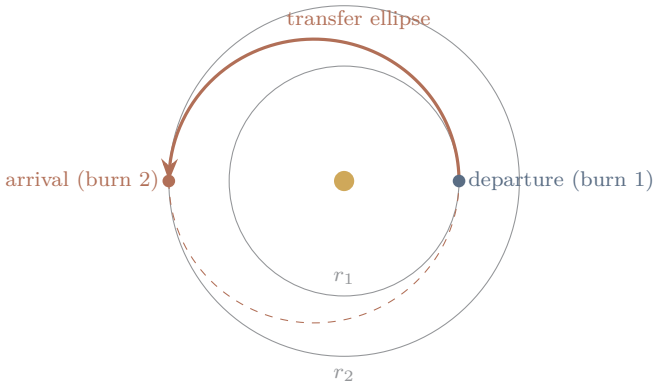


Figure 5.6: A Hohmann transfer from an inner circular orbit to an outer one, drawn to scale for the Earth and Mars.

Example 5.5.1 (To Mars)**.** A spacecraft travels from the Earth's orbit (1.000 AU) to that of Mars (1.524 AU) on a Hohmann transfer. Find the travel time and the two changes of speed. Ignore the gravity of the planets themselves.

Solution. The transfer ellipse has $a_{\mathrm{t}} = (1.000 + 1.524)/2 = 1.262\,\mathrm{AU}$ and period $1.262^{3/2} = 1.418$ years. The trip takes half of that, 0.709 years or 259 days.

Speeds from Equation (5.10): at departure the spacecraft needs

$$v = 29.8\sqrt{\frac{2}{1.000} - \frac{1}{1.262}} = 32.7\,\mathrm{km\,s^{-1}},$$

which is $2.9\,\mathrm{km\,s^{-1}}$ more than the Earth's own $29.8\,\mathrm{km\,s^{-1}}$. It arrives at aphelion with $29.8\sqrt{2/1.524 - 1/1.262} = 21.5\,\mathrm{km\,s^{-1}}$, while Mars moves at $29.8/\sqrt{1.524} = 24.1\,\mathrm{km\,s^{-1}}$, so a second increase of $2.6\,\mathrm{km\,s^{-1}}$ is needed. Notice that the spacecraft must speed up twice, yet ends up moving more slowly than it started: it has traded kinetic energy for potential energy.

5.6 Tides and eclipses

Tides

The Moon pulls slightly harder on the near side of the Earth than on its centre, and slightly less hard on the far side. Relative to the centre, the near side is pulled towards the Moon and the far side is left behind, so the oceans bulge on *both* sides and most coasts have two high tides a day. The stretching force is the difference in gravity across the Earth, which for a body of mass M at distance d is proportional to

$$F_{\mathrm{tide}} \propto \frac{M}{d^3}. \tag{5.12}$$

The inverse cube explains why the Moon raises larger tides than the Sun although the Sun's pull on the Earth is far stronger: $(M_\odot/d_\odot^3)/(M_{\mathrm{Moon}}/d_{\mathrm{Moon}}^3) = 0.46$.

When the Sun, Earth and Moon are in line, at new and full moon, the two effects add (*spring tides*); at the quarter phases they partly cancel (*neap tides*).

Eclipses

A **solar eclipse** happens at new moon, when the Moon passes between the Sun and the Earth; a **lunar eclipse** happens at full moon, when the Moon passes through the Earth's shadow. Eclipses do not occur every month because the Moon's orbit is tilted by about 5° to the ecliptic, so at most new and full moons the Moon passes above or below the Sun–Earth line. An eclipse requires the new or full moon to occur close to one of the two points where the Moon's orbit crosses the ecliptic, which happens during two 'eclipse seasons' each year.

5.7 Summary

Greatest elongation	$\sin \epsilon_{\max} = a\,[\mathrm{AU}]$
Synodic period	$1/S = \lvert 1/P - 1/E \rvert$
Ellipse	$r_p = a(1-e), \quad r_a = a(1+e)$
Second law	$r_p v_p = r_a v_a$
Third law	$T^2 = 4\pi^2 a^3/GM$
Third law, around the Sun	$T^2 = a^3 \quad (\mathrm{yr}, \mathrm{AU})$
Orbital energy	$E = -GMm/2a$
Vis-viva	$v^2 = GM(2/r - 1/a)$
Circular and escape speeds	$v_\mathrm{c} = \sqrt{GM/r}, \quad v_\mathrm{esc} = \sqrt{2}\, v_\mathrm{c}$
Hohmann transfer	$a_\mathrm{t} = (r_1 + r_2)/2$, time $= T_\mathrm{t}/2$
Tidal force	$\propto M/d^3$

Table 5.2: Formulas of Chapter 5.

PART II

Practice

CHAPTER 6

Practice Problems with Solutions

The problems below are grouped by the chapter they belong to. Many are taken or adapted from the short theoretical questions of the International Olympiad on Astronomy and Astrophysics (IOAA) and from the first round of the USA Astronomy and Astrophysics Olympiad (USAAAO); the source is shown beside the title. Try each problem before reading its solution, and write your own solutions the way a marker would want to see them: state the formula, substitute with units, and give the answer to a sensible number of significant figures.

Unless a problem says otherwise, use $m_\odot = -26.74$, $M_\odot = 4.83$, $L_\odot = 3.828 \times 10^{26}\,\mathrm{W}$, $R_\odot = 6.957 \times 10^{8}\,\mathrm{m}$, $T_\odot = 5772\,\mathrm{K}$, $1\,\mathrm{AU} = 1.496 \times 10^{11}\,\mathrm{m}$, $1\,\mathrm{pc} = 3.086 \times 10^{16}\,\mathrm{m}$ and $c = 3.00 \times 10^{5}\,\mathrm{km\,s^{-1}}$.

6.1 Starlight

Problem 6.1 (Solar flux and the solar constant)**.** The nominal luminosity of the Sun is defined by the IAU as $3.828 \times 10^{26}\,\mathrm{W}$. Assuming the Earth's orbit is a circle of radius $1\,\mathrm{AU} = 1.496 \times 10^{11}\,\mathrm{m}$, calculate the solar flux received at the top of the Earth's atmosphere (the solar constant). Then find the flux at Mercury (0.387 AU), Jupiter (5.20 AU) and Neptune (30.1 AU).

Solution. The flux at distance r from a source that radiates equally in all directions is $b = L/4\pi r^2$:

$$b_\oplus = \frac{3.828 \times 10^{26}}{4\pi\,(1.496 \times 10^{11})^2} = \frac{3.828 \times 10^{26}}{2.812 \times 10^{23}} = 1361\,\mathrm{W\,m^{-2}}.$$

For the other planets use the inverse-square law, $b = 1361/r^2$ with r in AU: $9090\,\mathrm{W\,m^{-2}}$ at Mercury, $50\,\mathrm{W\,m^{-2}}$ at Jupiter and $1.5\,\mathrm{W\,m^{-2}}$ at Neptune (Figure 6.1). On a logarithmic plot the inverse-square law is a straight line of slope -2.

Problem 6.2 (Stellar parallax and distance)**.** The annual parallax of a star is measured to be $0.40''$. What is its distance in parsecs and in metres?

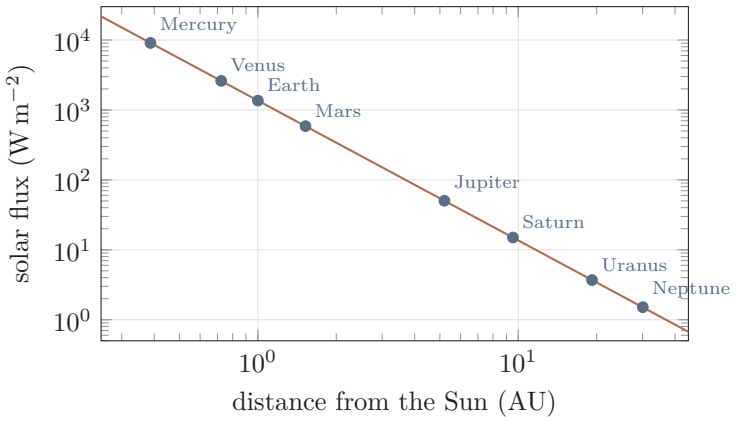


Figure 6.1: The flux of sunlight at the planets. Both axes are logarithmic.

Solution. $d = 1/p = 1/0.40 = 2.5\,\text{pc}$. In SI units, $2.5 \times 3.086 \times 10^{16} = 7.7 \times 10^{16}\,\text{m}$. The same result follows from the definition: $p = 0.40'' = 1.94 \times 10^{-6}\,\text{rad}$, and $d = 1\,\text{AU}/p = 1.496 \times 10^{11}/1.94 \times 10^{-6} = 7.7 \times 10^{16}\,\text{m}$.

Problem 6.3 (Comparing the distance and brightness of two stars)**.** Star A has apparent magnitude 3.5 and parallax $0.05''$. Star B has apparent magnitude 2.0 and parallax $0.02''$.

(a) Which star is closer to the Earth, and by how much?

(b) Which star is really the more luminous, and by what factor?

Solution.

(a) $d_A = 1/0.05 = 20\,\text{pc}$ and $d_B = 1/0.02 = 50\,\text{pc}$. Star A is closer, by $30\,\text{pc}$.

(b) Compare absolute magnitudes:

$$M_A = 3.5 - 5\log_{10} 20 + 5 = 1.99, \qquad M_B = 2.0 - 5\log_{10} 50 + 5 = -1.49.$$

Star B is the more luminous, by 3.48 magnitudes, a factor of $10^{0.4\times 3.48} = 25$. It looks brighter than A even though it is two and a half times farther away. (Check: B delivers $10^{0.4\times 1.5} = 3.98$ times the flux of A from 2.5 times the distance, so its luminosity is $3.98 \times 2.5^2 = 25$ times greater.)

Problem 6.4 (Distance and luminosity from parallax and magnitude)**.** A star has apparent magnitude $m_V = 2.5$ and parallax $0.05''$. Calculate its distance, its absolute magnitude and its luminosity in solar units.

Solution. $d = 1/0.05 = 20\,\mathrm{pc}$. The absolute magnitude is $M_V = 2.5 - 5\log_{10} 20 + 5 = 0.99$, and

$$\frac{L}{L_\odot} = 10^{0.4\,(M_\odot - M_V)} = 10^{0.4\,(4.83-0.99)} = 10^{1.534} = 34.$$

Problem 6.5 (Flux from a star)**.** A star has apparent magnitude $m = 2.00$. The solar constant is $1361\,\mathrm{W\,m^{-2}}$. What flux does the Earth receive from the star?

Solution. Compare the star with the Sun:

$$\frac{b}{b_\odot} = 10^{-0.4\,(2.00-(-26.74))} = 10^{-11.50} = 3.19 \times 10^{-12},$$

so $b = 1361 \times 3.19 \times 10^{-12} = 4.3 \times 10^{-9}\,\mathrm{W\,m^{-2}}$.

Problem 6.6 (Distance of an RR Lyrae star [adapted from USAAAO 2016, First Round, Q1])**.** RR Lyrae variables all have absolute magnitude close to $M = 0.75$. One is observed with apparent magnitude 16.0. How far away is it? Ignore extinction.

Solution. The distance modulus is $m - M = 15.25$, so $5\log_{10} d = 20.25$ and $d = 10^{4.05} = 1.12 \times 10^4\,\mathrm{pc} = 11.2\,\mathrm{kpc}$.

Problem 6.7 (Distance, absolute magnitude and radius)**.** A star has apparent magnitude $m_V = 3.50$, parallax $0.030''$ and surface temperature $4500\,\mathrm{K}$.

(a) Find its distance.

(b) Find its absolute magnitude.

(c) Estimate its luminosity and radius in solar units, treating the visual magnitude as bolometric. What kind of star is it?

Solution.

(a) $d = 1/0.030 = 33.3\,\mathrm{pc}$.

(b) $M_V = 3.50 - 5\log_{10}(33.3) + 5 = 0.89$.

(c) $L/L_\odot = 10^{0.4(4.83-0.89)} = 38$. From $L \propto R^2T^4$,

$$\frac{R}{R_\odot} = \sqrt{\frac{L}{L_\odot}}\left(\frac{T_\odot}{T}\right)^2 = \sqrt{38} \times \left(\frac{5772}{4500}\right)^2 = 10.$$

A cool star ten times the size of the Sun is a red giant.

Problem 6.8 (Two main-sequence stars [adapted from USAAAO 2016, First Round, Q16])**.** Star A has surface temperature 10 000 K and star B 5700 K. Their radii are in the ratio $R_A : R_B = 3 : 2$. Find the difference of their absolute magnitudes, $M_B - M_A$.

Solution.

$$\frac{L_A}{L_B} = \left(\frac{3}{2}\right)^2 \left(\frac{10\,000}{5700}\right)^4 = 2.25 \times 9.47 = 21.3,$$

so $M_B - M_A = 2.5 \log_{10}(21.3) = 3.3$. B is the fainter star, so its magnitude is the larger one.

Problem 6.9 (A star cluster)**.** A distant cluster contains 100 stars, each of apparent magnitude 10.0, too close together to be resolved. What is the apparent magnitude of the cluster as a whole?

Solution. The total flux is 100 times that of one star, which is exactly 5 magnitudes: $m = 10.0 - 2.5 \log_{10} 100 = 5.0$. The cluster is just visible to the naked eye although none of its stars is.

Problem 6.10 (Main-sequence fitting with extinction)**.** Fitting the main sequence of a cluster gives an apparent distance modulus $m_V - M_V = 12.0$. The extinction towards the cluster is $A_V = 1.5$. Find the distance, and the distance that would be obtained if extinction were ignored.

Solution. With extinction, $5 \log_{10} d - 5 = 12.0 - 1.5 = 10.5$, so $d = 10^{3.1} = 1260$ pc. Ignoring extinction would give $d = 10^{3.4} = 2510$ pc, twice too far.

Problem 6.11 (Distance from luminosity and flux [adapted from IOAA 2007, short question 1.10])**.** A star has luminosity $0.40\,L_\odot$, and the flux received from it at the Earth is 6.23×10^{-14} W m^{-2}. How far away is it? Ignore extinction.

Solution. From $b = L/4\pi d^2$,

$$d = \sqrt{\frac{L}{4\pi b}} = \sqrt{\frac{0.40 \times 3.828 \times 10^{26}}{4\pi \times 6.23 \times 10^{-14}}} = 1.40 \times 10^{19}\,\text{m} = 453\,\text{pc}.$$

Problem 6.12 (A pulsating star [IOAA 2010, short problem 10])**.** The brightness of a Cepheid variable changes by 2 magnitudes between maximum and minimum light. Its effective temperature is 6000 K at maximum and 5000 K at minimum. Find the ratio of its radius at maximum light to its radius at minimum light.

Solution. Two magnitudes is a luminosity ratio $L_{\max}/L_{\min} = 10^{0.4\times 2} = 6.31$. Since $L \propto R^2T^4$,

$$\frac{R_{\max}}{R_{\min}} = \sqrt{\frac{L_{\max}}{L_{\min}}}\left(\frac{T_{\min}}{T_{\max}}\right)^2 = \sqrt{6.31}\times\left(\frac{5000}{6000}\right)^2 = 2.51\times 0.694 = 1.7.$$

Both the size and the temperature of the star change; the change of temperature alone would account for a factor of only $(6/5)^4 = 2.1$ in luminosity.

Problem 6.13 (Temperature of a planet [adapted from USAAAO 2016, First Round, Q22])**.** A planet orbits 1.5 AU from a star of surface temperature 6500 K and radius $1.2\,R_\odot$. The planet reflects 10% of the light that falls on it and rotates quickly. Estimate its equilibrium temperature.

Solution. In equilibrium the planet radiates as much power as it absorbs. The flux at the planet is $b = L/4\pi d^2 = \sigma T_\star^4 R_\star^2/d^2$. A planet of radius R_p and albedo A intercepts sunlight over its cross-section πR_p^2 and absorbs a fraction $1-A$; a rapidly rotating planet radiates from its whole surface $4\pi R_p^2$:

$$(1-A)\,\pi R_p^2\,\frac{\sigma T_\star^4 R_\star^2}{d^2} = 4\pi R_p^2\,\sigma T_p^4 \quad\Rightarrow\quad T_p = T_\star\sqrt{\frac{R_\star}{2d}}\,(1-A)^{1/4}.$$

The radius of the planet cancels. With $R_\star = 1.2\times 6.957\times 10^8 = 8.35\times 10^8$ m and $d = 2.244\times 10^{11}$ m,

$$T_p = 6500\times\sqrt{\frac{8.35\times 10^8}{4.488\times 10^{11}}}\times 0.90^{1/4} = 6500\times 0.0431\times 0.974 = 273\,\text{K}.$$

6.2 Telescopes

Problem 6.14 (Telescope limiting magnitude)**.** Determine the theoretical limiting magnitude of an 8-inch (203 mm) telescope. Assume the dark-adapted eye has a pupil diameter of 6 mm and a naked-eye limiting magnitude of 6.5. Ignore atmospheric extinction and losses in the optics.

Solution. The light gathered is proportional to the area of the aperture, so the gain in magnitudes is $2.5\log_{10}(D_t/D_e)^2 = 5\log_{10}(D_t/D_e)$:

$$m_t = 6.5 + 5\log_{10}\frac{203}{6} = 6.5 + 5(1.529) = 14.1.$$

Figure 6.2 shows how the limit grows with aperture: every factor of 10 in diameter gains 5 magnitudes.

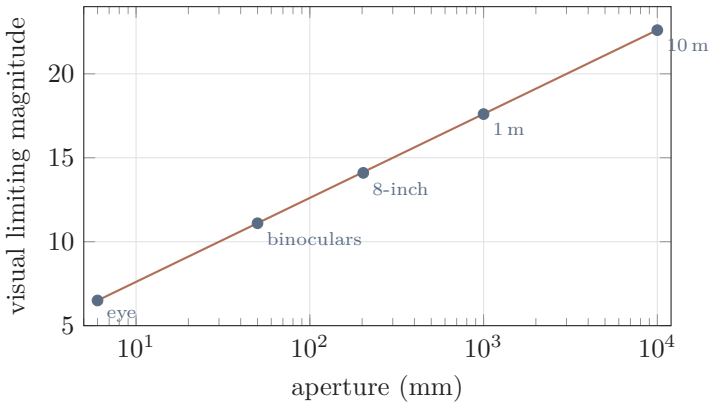


Figure 6.2: Visual limiting magnitude against aperture, for a naked-eye limit of 6.5 and a 6 mm pupil.

Problem 6.15 (Telescope magnification)**.** A telescope has an objective of focal length 1000 mm and an eyepiece of focal length 20 mm.

(a) Determine the magnification.

(b) The telescope is pointed at a road sign 0.50 m wide and 2000 m away. What angle does the sign subtend to the naked eye, and what angle does it appear to subtend in the eyepiece?

(c) How far behind the focal plane for stars is the image of the sign formed?

Solution.

(a) $M = f_{\text{obj}}/f_{\text{e}} = 1000/20 = 50$.

(b) To the eye the sign subtends $0.50/2000 = 2.5 \times 10^{-4}$ rad $= 52''$, about the limit of what the eye can resolve. In the eyepiece it appears 50 times larger: $0.0125\,\text{rad} = 0.72°$, larger than the full Moon.

(c) An object at distance u is imaged at distance v given by $1/v = 1/f - 1/u$. With $f = 1.000$ m and $u = 2000$ m, $v = 1.0005$ m: the image lies 0.5 mm behind the focal plane, so the eyepiece must be racked out by that amount. Because the object distance is so much larger than the focal length, the magnification is still $f_{\text{obj}}/f_{\text{e}}$ to a very good approximation.

Problem 6.16 (Imaging Jupiter with a telescope)**.** A telescope of focal length 2000 mm and aperture 200 mm is fitted with a camera whose pixel scale is $1.0''$ per pixel. Jupiter has an angular diameter of $50''$.

(a) What is the physical size of each pixel?

(b) How many pixels span Jupiter's disc, and how wide is the image of the planet on the sensor?

(c) What is the diffraction limit of the telescope at 550 nm? Is the image well sampled?

Solution.

(a) The plate scale is $206\,265/2000 = 103''$ per mm, so a pixel covering $1.0''$ is $1/103\,\text{mm} = 9.7\,\mu\text{m}$ across.

(b) Jupiter spans 50 pixels, or $50 \times 9.7\,\mu\text{m} = 0.49\,\text{mm}$.

(c) $\theta = 1.22 \times 5.5 \times 10^{-7}/0.200 = 3.4 \times 10^{-6}\,\text{rad} = 0.69''$. The pixels ($1.0''$) are larger than the finest detail the telescope can deliver, so the image is undersampled; pixels of about $0.35''$, half the diffraction limit, would be needed to record everything the optics can show.

Problem 6.17 (Imaging the full Moon)**.** A camera with $5.0\,\mu$m pixels is placed at the focus of a telescope of focal length 1200 mm.

(a) Find the plate scale in arcseconds per millimetre.

(b) How large is the image of the full Moon (angular diameter $30'$)?

(c) How many pixels span the Moon's diameter?

Solution.

(a) Plate scale $= 206\,265/1200 = 172''$ per mm.

(b) The Moon is $1800''$ across, so its image is $1800/172 = 10.5$ mm in diameter.

(c) $10.5\,\text{mm}/0.0050\,\text{mm} = 2100$ pixels. Equivalently, the pixel scale is $0.86''$ and $1800/0.86 = 2100$.

Problem 6.18 (Jupiter at opposition)**.** At opposition Jupiter is 4.20 AU from the Earth and has an angular diameter of $46.8''$. It is observed with a telescope of focal length 1000 mm.

(a) Find Jupiter's diameter in kilometres.

(b) How large is its image at the focus of the telescope?

(c) With an eyepiece of focal length 10 mm, how large does Jupiter appear? Compare with the Moon seen by the naked eye.

Solution.

(a) $D = d\,\theta = (4.20 \times 1.496 \times 10^{8}\,\text{km}) \times 46.8/206\,265 = 1.43 \times 10^{5}\,\text{km}$.

(b) The plate scale is $206\,265/1000 = 206''$ per mm, so the image is $46.8/206 = 0.23$ mm across.

(c) The magnification is $1000/10 = 100$, so Jupiter appears $4680'' = 1.3°$ across, more than twice the apparent diameter of the Moon.

The distance needed in part (a) is the distance from the *Earth*, not from the Sun.

Problem 6.19 (Counting photons [IOAA 2010, short problem])**.** A star of magnitude $I = 22.0$ is observed with an 8 m telescope through an I-band filter of effective wavelength 800 nm and width 240 nm. A star of magnitude $I = 0$ gives a flux of $0.83 \times 10^{-11}\,\mathrm{W\,m^{-2}\,nm^{-1}}$ in this band. The detector records 40% of the photons that reach it. How many photons from the star does it record per second?

Solution. Flux per nanometre from the star: $0.83 \times 10^{-11} \times 10^{-0.4\times 22.0} = 1.32 \times 10^{-20}\,\mathrm{W\,m^{-2}\,nm^{-1}}$.

Energy of one photon: $hc/\lambda = (6.626 \times 10^{-34})(2.998 \times 10^{8})/(8.0 \times 10^{-7}) = 2.48 \times 10^{-19}\,\mathrm{J}$, so the star delivers $1.32 \times 10^{-20}/2.48 \times 10^{-19} = 0.0530$ photons per second per square metre per nanometre.

The mirror has area $\pi \times 4^2 = 50.3\,\mathrm{m^2}$, the filter passes 240 nm, and 40% of the photons are recorded:

$$N = 0.0530 \times 50.3 \times 240 \times 0.40 = 2.6 \times 10^{2} \text{ photons per second.}$$

Even one of the largest telescopes in the world collects only a few hundred photons a second from such a star, which is why faint objects need long exposures.

Problem 6.20 (Detecting an Earth [IOAA 2010, short problem 7])**.** An Earth-sized planet passes in front of a Sun-like star. By what fraction does the brightness of the star fall? Repeat for a planet the size of Jupiter (radius 69 900 km), and comment on what precision a detector needs in each case.

Solution. The planet hides a fraction of the star's disc equal to the ratio of the two areas:

$$\frac{\Delta b}{b} = \left(\frac{R_p}{R_\star}\right)^2 = \left(\frac{6371}{695\,700}\right)^2 = 8.4 \times 10^{-5}$$

for the Earth, and $(69\,900/695\,700)^2 = 0.010$ for Jupiter. A Jupiter produces a dip of one per cent, about 0.01 magnitudes, which a good ground-based telescope can measure. An Earth produces a dip of less than one part in ten thousand, which requires the stability of a space telescope.

6.3 Stars, galaxies and cosmology

Problem 6.21 (Stellar lifetimes)**.** Estimate the main-sequence lifetime of a $10\,M_\odot$ star. Assume the mass–luminosity relation $L \propto M^{3.5}$ and a main-sequence lifetime of 10^{10} years for the Sun.

Solution. The lifetime is proportional to the fuel supply (M) divided by the rate at which it is used (L):

$$t_{\rm MS} \propto \frac{M}{L} \propto \frac{M}{M^{3.5}} = M^{-2.5}, \qquad \text{so} \qquad \frac{t_\star}{t_\odot} = \left(\frac{M_\star}{M_\odot}\right)^{-2.5}.$$

For $10\,M_\odot$, $10^{-2.5} = 1/316$ and $t_\star = 10^{10}/316 = 3.2 \times 10^7$ years, about 32 million years (Figure 6.3).

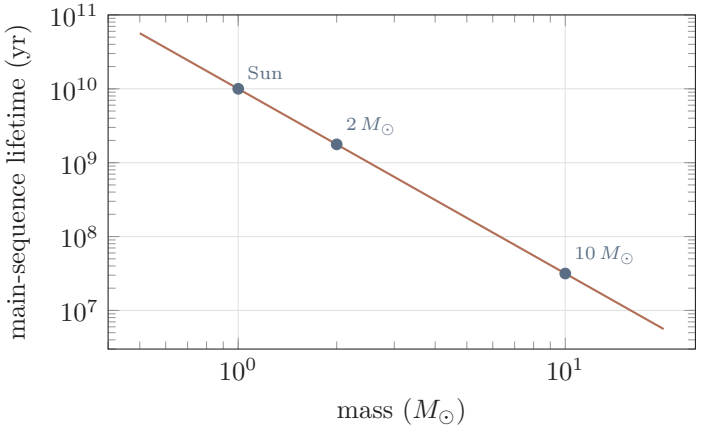


Figure 6.3: Main-sequence lifetime against mass, from $t \propto M^{-2.5}$.

Problem 6.22 (Redshift and Hubble's law)**.** The H and K lines of ionised calcium have rest wavelengths 3968.5 Å and 3933.7 Å. In a galaxy of the cluster Abell 2065 they are observed at 4255.0 Å and 4217.6 Å.

(a) Calculate the redshift of the galaxy.

(b) Determine its radial velocity and distance, using $H_0 = 73\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. Discuss whether a relativistic calculation is required.

Solution.

(a)

$$z_{\rm H} = \frac{4255.0 - 3968.5}{3968.5} = 0.07219, \qquad z_{\rm K} = \frac{4217.6 - 3933.7}{3933.7} = 0.07217.$$

The two lines agree, as they must for a true redshift; adopt $z = 0.0722$.

(b) With the low-redshift approximation, $v = cz = 2.16 \times 10^4\,\mathrm{km\,s^{-1}}$ and $d = v/H_0 = 297\,\mathrm{Mpc}$.

With the relativistic Doppler formula,

$$\frac{v}{c} = \frac{(1+z)^2 - 1}{(1+z)^2 + 1} = \frac{0.1496}{2.1496} = 0.0696,$$

so $v = 2.09 \times 10^4\,\mathrm{km\,s^{-1}}$ and $d = 286\,\mathrm{Mpc}$. The two answers differ by about 4%, which is roughly $z/2$. At $z < 0.1$ either method is normally accepted, and $v = cz$ is the usual choice; say which one you have used. The difference is a reminder that at larger redshifts neither is exact.

Problem 6.23 (A high-redshift quasar)**.** The quasar SDSS 1030+0524 shows the Lyman-α line of hydrogen (rest wavelength 121.6 nm) at an observed wavelength of 885.2 nm.

(a) Find the redshift.

(b) In which part of the spectrum was the light emitted, and in which part is it observed?

(c) Explain why $v = cz$ cannot be used here. What speed does the relativistic Doppler formula give?

(d) Use that speed in Hubble's law with $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. Why is the result not the true distance?

Solution.

(a) $z = (885.2 - 121.6)/121.6 = 6.28$.

(b) The light was emitted in the far ultraviolet and arrives in the near infrared. Every wavelength has been stretched by the factor $1 + z = 7.28$.

(c) $v = cz$ would give more than six times the speed of light. The relativistic formula gives

$$\frac{v}{c} = \frac{(1+z)^2 - 1}{(1+z)^2 + 1} = \frac{52.0}{54.0} = 0.963.$$

(d) $d = 0.963 \times 3.00 \times 10^5/70 = 4.1 \times 10^3\,\mathrm{Mpc} = 4.1\,\mathrm{Gpc}$. This is only a formal exercise. Hubble's law $v = H_0 d$ with today's value of H_0 holds for nearby galaxies; over the time this light has been travelling the expansion rate has changed greatly. A proper calculation with the standard cosmological model ($H_0 = 70$, matter density 30% of critical) gives a present-day distance of about 8.4 Gpc and a light-travel time of 12.6 billion years: we see the quasar as it was less than a billion years after the Big Bang.

Problem 6.24 (Stars above the Galactic plane [IOAA 2010, short problem])**.** The number density of stars in the disc of the Milky Way falls off with height z

above the mid-plane as $n(z) = n_0\, e^{-z/h}$, with scale height $h = 300\,\mathrm{pc}$. Find the density of stars at heights of 0.5 kpc and 1.5 kpc, as fractions of the density in the mid-plane.

Solution.

$$\frac{n(0.5\,\mathrm{kpc})}{n_0} = e^{-500/300} = 0.19, \qquad \frac{n(1.5\,\mathrm{kpc})}{n_0} = e^{-1500/300} = e^{-5} = 0.0067.$$

Three times higher, the density is nearly thirty times lower: the stellar disc is very thin compared with its 30 kpc diameter.

Problem 6.25 (Looking back in time)**.** A galaxy is 100 Mpc away. Taking $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, find its recession speed and redshift, the time its light has taken to reach us, and that time as a fraction of the Hubble time.

Solution. $v = H_0 d = 7000\,\mathrm{km\,s^{-1}}$ and $z = v/c = 0.023$. One megaparsec is 3.26 million light-years, so the light has travelled for 326 million years. The Hubble time is $1/H_0 = 14.0$ billion years, so we see the galaxy as it was when the universe was about 2.3% younger than it is now. For nearby galaxies the look-back fraction is simply equal to the redshift.

Problem 6.26 (The black hole at the centre of the Milky Way)**.** Sagittarius A* has a mass of $4.3 \times 10^6\, M_\odot$ and is 8.2 kpc away. Find its Schwarzschild radius in astronomical units, and the angular diameter of a sphere of that radius seen from the Earth.

Solution. $R_s = 2.95\,\mathrm{km} \times 4.3 \times 10^6 = 1.27 \times 10^7\,\mathrm{km} = 0.085\,\mathrm{AU}$, about a fifth of the size of Mercury's orbit. One AU at one parsec subtends one arcsecond, so the diameter $2R_s = 0.170\,\mathrm{AU}$ at 8200 pc subtends

$$\theta = \frac{0.170}{8200}\ \text{arcsec} = 2.1 \times 10^{-5}\ \text{arcsec} = 21\ \text{micro-arcseconds}.$$

(The dark shadow actually observed is about two and a half times larger, because the black hole bends light around itself.)

Problem 6.27 (A spectroscopic binary)**.** The spectral lines of the two stars of an eclipsing binary shift back and forth with a period of 10.0 days. The orbits are circular, and the orbital speeds are $50\,\mathrm{km\,s^{-1}}$ for star 1 and $100\,\mathrm{km\,s^{-1}}$ for star 2. Find the separation of the stars and the mass of each.

Solution. Because the system eclipses, we see the orbit edge-on and the measured speeds are the true orbital speeds. Each star travels the circumference of its orbit in one period, so

$$a_1 = \frac{v_1 P}{2\pi} = \frac{(5.0\times10^4)(8.64\times10^5)}{2\pi} = 6.88\times10^9\,\mathrm{m},$$

and $a_2 = 2a_1 = 1.38\times10^{10}\,\mathrm{m}$. The separation is $a = a_1 + a_2 = 2.06\times10^{10}\,\mathrm{m} = 0.138\,\mathrm{AU}$. With $P = 10.0/365.25 = 0.0274\,\mathrm{yr}$,

$$M_1 + M_2 = \frac{a^3}{P^2} = \frac{0.138^3}{0.0274^2} = 3.5\,M_\odot.$$

The mass ratio is $M_1/M_2 = v_2/v_1 = 2$, so $M_1 = 2.3\,M_\odot$ and $M_2 = 1.2\,M_\odot$. The slower star is the heavier one.

Problem 6.28 (Dark matter in a spiral galaxy)**.** The rotation curve of a spiral galaxy is flat at $200\,\mathrm{km\,s^{-1}}$ out to $30\,\mathrm{kpc}$ from the centre. The galaxy's luminosity is $2.0\times10^{10}\,L_\odot$. Find the mass inside $30\,\mathrm{kpc}$ and the mass-to-light ratio. Comment.

Solution.

$$M = \frac{v^2 r}{G} = \frac{(2.0\times10^5)^2(30\,000\times3.086\times10^{16})}{6.674\times10^{-11}}$$
$$= 5.5\times10^{41}\,\mathrm{kg} = 2.8\times10^{11}\,M_\odot.$$

The mass-to-light ratio is $2.8\times10^{11}/2.0\times10^{10} = 14$ in solar units. A population of ordinary stars has a ratio of only a few, so most of the mass of this galaxy is not in stars: it is dark matter.

6.4 Spherical astronomy

Problem 6.29 (Spherical trigonometry)**.** On a sphere of unit radius, each triangle below is defined by three of its sides (a, b, c) and angles (A, B, C). Find the unknown elements and the area (the spherical excess) of each.

(a) $a = 34°46'$, $b = 27°22'$, $C = 72°31'$

(b) $b = 98°18'$, $C = 24°49'$, $A = 68°36'$

(c) $a = 14°03'$, $b = 53°32'$, $c = 62°14'$

(d) $A = 83°32'$, $B = 102°38'$, $C = 34°44'$

Solution. Before starting, check that a triangle with the given data can exist: any two sides must together exceed the third, and the three angles must add up to more than 180°.

(a) *Two sides and the included angle.* This triangle is solved step by step in the first worked example of Chapter 4: $c = 36°04'$, $A = 67°29'$, $B = 48°08'$, and the excess is $8°08'$, an area of $0.142\,\mathrm{sr}$.

(b) *Two angles and the side between them.* Use the polar formula:

$$\begin{aligned}\cos B &= -\cos A\cos C + \sin A\sin C\cos b \\ &= -(0.3649)(0.9077) + (0.9311)(0.4197)(-0.1444) = -0.3876,\end{aligned}$$

so $B = 112°48'$. The remaining sides follow from the polar formula rearranged, for example $\cos a = (\cos A + \cos B\cos C)/(\sin B\sin C)$, which gives $a = 88°04'$, and similarly $c = 26°47'$. The excess is $26°13'$, an area of $0.458\,\mathrm{sr}$.

(c) *Three sides.* Use the cosine formula for each angle:

$$\cos A = \frac{\cos a - \cos b\cos c}{\sin b\sin c} = \frac{0.9701 - (0.5944)(0.4659)}{(0.8042)(0.8848)} = 0.9741,$$

so $A = 13°04'$. In the same way $B = 48°28'$ and $C = 124°33'$. The excess is $6°05'$, an area of $0.106\,\mathrm{sr}$.

(d) *Three angles.* Use the polar formula for each side:

$$\cos a = \frac{\cos A + \cos B\cos C}{\sin B\sin C} = \frac{0.1126 + (-0.2187)(0.8218)}{(0.9758)(0.5697)} = -0.1207,$$

so $a = 96°56'$. In the same way $b = 102°53'$ and $c = 34°42'$. The excess is $40°54'$, an area of $0.714\,\mathrm{sr}$. Unlike plane triangles, a spherical triangle is fixed completely by its three angles.

Problem 6.30 (Two cities on the same parallel)**.** Two cities lie at latitude $43°39'$ N and differ in longitude by $127°22'$. Take the radius of the Earth to be $6371\,\mathrm{km}$.

(a) Find the distance between them along the parallel of latitude.

(b) Find the great-circle distance between them.

(c) Find the highest latitude reached by the great-circle route.

Solution.

(a) The parallel is a small circle of radius $R\cos\phi$. With $\Delta l = 127.37° = 2.223\,\mathrm{rad}$, the distance is $6371 \times \cos 43.65° \times 2.223 = 10\,250\,\mathrm{km}$.

(b) From the cosine formula applied to the triangle formed by the two cities and the north pole,

$$\cos d = \sin^2\phi + \cos^2\phi\cos\Delta l = 0.4764 + (0.5236)(-0.6069) = 0.1586,$$

so $d = 80.87° = 1.411\,\mathrm{rad}$ and the distance is $6371 \times 1.411 = 8990\,\mathrm{km}$, some $1260\,\mathrm{km}$ shorter than the route along the parallel.

(c) By symmetry the route is highest midway in longitude. In the right-angled spherical triangle formed by the pole P, one city A and the highest point V, the angle at the pole is $\Delta l/2 = 63°41'$, the side PA is $90° - \phi = 46°21'$, and the angle at V is $90°$. The four-parts formula gives

$$\tan PV = \cos(63°41')\tan(46°21') = 0.465 \;\Rightarrow\; PV = 24.9°,$$

so the highest latitude is $90° - 24.9° = 65.1°$ N. This is why flights between cities at middle latitudes pass so far north.

Problem 6.31 (Celestial coordinates: Rigel in Toronto)**.** An observer in Toronto ($\phi = 43.65°$ N) is tracking Rigel ($\alpha = 5^{\mathrm{h}}14^{\mathrm{m}}$, $\delta = -8°12'$).

(a) Determine the maximum altitude of Rigel in Toronto's sky.

(b) Find the azimuth of Rigel at rising and at setting.

(c) Find the azimuth and hour angle when Rigel's altitude is $8°$.

(d) Find the altitude and azimuth when the hour angle is $1^{\mathrm{h}}53^{\mathrm{m}}$.

(e) Determine the angle that Rigel's path makes with the horizon at rising or setting.

Solution.

(a) $h_{\max} = 90° - \phi + \delta = 90° - 43.65° - 8.20° = 38.2°$.

(b) $\cos A_0 = \sin\delta/\cos\phi = -0.1426/0.7236 = -0.197$, so Rigel rises at $A = 101.4°$ (south of east) and sets at $360° - 101.4° = 258.6°$.

(c) From $\sin h = \sin\phi\sin\delta + \cos\phi\cos\delta\cos H$,

$$\cos H = \frac{\sin h - \sin\phi\sin\delta}{\cos\phi\cos\delta} = \frac{0.1392 + 0.0984}{(0.7236)(0.9898)} = 0.332,$$

so $H = \pm 70.6° = \pm 4^{\mathrm{h}}42^{\mathrm{m}}$: the star is at this altitude $4^{\mathrm{h}}42^{\mathrm{m}}$ before and after it crosses the meridian. For the azimuth,

$$\cos A = \frac{\sin\delta - \sin\phi\sin h}{\cos\phi\cos h} = \frac{-0.1426 - (0.6903)(0.1392)}{(0.7236)(0.9903)} = -0.333,$$

so $A = 109.5°$ while rising and $360° - 109.5° = 250.5°$ while setting.

(d) $H = 1^{\mathrm{h}}53^{\mathrm{m}} = 28.25°$, and

$$\sin h = (0.6903)(-0.1426) + (0.7236)(0.9898)(0.8809) = 0.5325,$$

so $h = 32.2°$. Then

$$\cos A = \frac{-0.1426 - (0.6903)(0.5325)}{(0.7236)(0.8464)} = -0.833,$$

which gives $146.4°$ or $213.6°$. The hour angle is positive, so the star is west of the meridian and $A = 213.6°$.

(e) In the triangle PZX at the moment of rising, the angle at the star between the directions to the zenith and to the pole is q, with $\cos q = \sin\phi/\cos\delta = 0.6903/0.9898 = 0.697$, so $q = 45.8°$. The star moves at right angles to the direction of the pole, and the horizon is at right angles to the direction of the zenith, so the path meets the horizon at the same angle, 45.8°. (For a star on the celestial equator the angle is $90° - \phi$.)

Figure 6.4 shows the altitude of Rigel through a whole day, together with that of Vega from the problem after next.

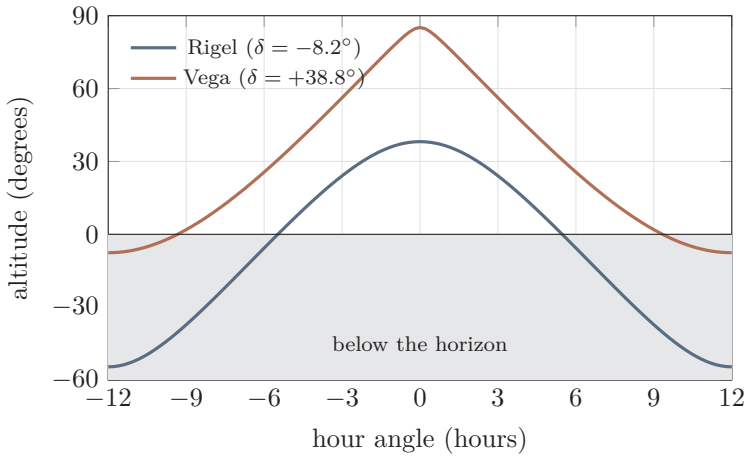


Figure 6.4: Altitude against hour angle for Rigel and Vega seen from Toronto. Rigel is above the horizon for less than twelve hours, Vega for nearly nineteen.

Problem 6.32 (Ecliptic coordinates of Rigel)**.** Find the ecliptic longitude and latitude of Rigel ($\alpha = 5^{\mathrm{h}}14^{\mathrm{m}}$, $\delta = -8°12'$), taking $\varepsilon = 23.44°$.

Solution. $\alpha = 5.233^{\mathrm{h}} \times 15 = 78.50°$ and $\delta = -8.20°$.

$$\begin{aligned}\sin\beta &= \sin\delta\cos\varepsilon - \cos\delta\sin\varepsilon\sin\alpha \\ &= (-0.1426)(0.9175) - (0.9898)(0.3978)(0.9799) = -0.5167,\end{aligned}$$

so $\beta = -31.1°$. For the longitude, $\cos\beta\cos\lambda = \cos\delta\cos\alpha$ gives

$$\cos\lambda = \frac{(0.9898)(0.1994)}{0.8562} = 0.2305 \;\Rightarrow\; \lambda = 76.7° \text{ or } 283.3°.$$

The right ascension is between 0^{h} and 6^{h}, so the star lies in the first quadrant of longitude: $\lambda = 76.7°$, $\beta = -31.1°$. Rigel is far south of the ecliptic, so the Sun, Moon and planets never pass near it.

Problem 6.33 (Vega's motion in Toronto's sky)**.** Vega has $\alpha = 18^{\mathrm{h}}36^{\mathrm{m}}$, $\delta = +38°47'$. An observer in Toronto ($\phi = 43.65°$ N) observes this star.

(a) Determine the hour angle of Vega at rising and setting.

(b) Find the azimuth of rising and setting.

(c) Determine its maximum altitude.

(d) Find the total time Vega is above the horizon. Is it circumpolar?

(e) Determine approximately the date on which Vega rises at the same time as the Sun.

Solution.

(a) $\cos H_0 = -\tan\phi\tan\delta = -(0.9540)(0.8034) = -0.7664$, so $H_0 = 140.0^\circ = 9^{\rm h}20^{\rm m}$. Vega rises at $H = -9^{\rm h}20^{\rm m}$ and sets at $H = +9^{\rm h}20^{\rm m}$. The cosine is negative because the star is in the same hemisphere as the observer.

(b) $\cos A_0 = \sin\delta/\cos\phi = 0.6263/0.7236 = 0.8656$, so it rises at $A = 30.0^\circ$, well to the north of east, and sets at 330.0°.

(c) $h_{\max} = 90^\circ - |\phi - \delta| = 90^\circ - 4.87^\circ = 85.1^\circ$, almost overhead.

(d) $2H_0 = 18^{\rm h}40^{\rm m}$ of each sidereal day. A star is circumpolar if $\delta > 90^\circ - \phi = 46.35^\circ$; Vega's declination is smaller, so it does set, though only for $5^{\rm h}20^{\rm m}$. It becomes circumpolar north of latitude 51.2°.

(e) Vega rises when the sidereal time is $\mathrm{LST} = \alpha - H_0 = 18^{\rm h}36^{\rm m} - 9^{\rm h}20^{\rm m} = 9^{\rm h}16^{\rm m}$. The Sun must rise at the same sidereal time, so we need $\alpha_\odot - H_{0\odot} = 9^{\rm h}16^{\rm m}$, where both the Sun's right ascension and its own rising hour angle depend on the date. Proceed by trial. In late October the Sun is about 37° of ecliptic longitude past the September equinox, $\lambda_\odot \approx 217^\circ$, which gives

$$\sin\delta_\odot = \sin\varepsilon\sin\lambda_\odot \;\Rightarrow\; \delta_\odot = -14.0^\circ, \qquad \alpha_\odot = 14^{\rm h}21^{\rm m},$$

and $\cos H_{0\odot} = -\tan(43.65^\circ)\tan(-14.0^\circ) = 0.238$, so $H_{0\odot} = 5^{\rm h}05^{\rm m}$. Then $\alpha_\odot - H_{0\odot} = 9^{\rm h}16^{\rm m}$, as required. The Sun moves about one degree a day and reaches this longitude some 38 days after the September equinox, so Vega and the Sun rise together at the very end of October. (Trying a date a week earlier or later changes $\alpha_\odot - H_{0\odot}$ by about half an hour, so the answer is good to a day or two.)

Problem 6.34 (Where on the horizon does a star rise?)**.** Show that a star of declination δ rises at an angle a north of the east point, where $\sin a = \sin\delta/\cos\phi$. Hence find the range of directions over which the Sun rises during the year in Toronto.

Solution. Setting $h = 0$ in $\sin\delta = \sin\phi\sin h + \cos\phi\cos h\cos A$ gives $\cos A = \sin\delta/\cos\phi$, with the azimuth A measured from north. The angle north of east is $a = 90^\circ - A$, and $\cos A = \sin(90^\circ - A) = \sin a$, which proves the

result. The angle a is called the *amplitude*: it is zero for a star on the equator, positive (north of east) for $\delta > 0$ and negative for $\delta < 0$.

For the Sun in Toronto, $\delta_\odot$ ranges between $\pm 23.44°$, so $\sin a = \pm 0.3978/0.7236 = \pm 0.550$ and $a = \pm 33.4°$. The sunrise point swings through $67°$ of the horizon between the solstices.

6.5 The solar system and celestial mechanics

Problem 6.35 (The orbit of Mars from its oppositions)**.** Successive oppositions of Mars are 779.9 days apart. Find the sidereal period of Mars and the semi-major axis of its orbit.

Solution. Mars is a superior planet, so $1/S = 1/E - 1/P$:

$$\frac{1}{P} = \frac{1}{365.256} - \frac{1}{779.9} = 1.4556 \times 10^{-3}\,\mathrm{d}^{-1} \;\Rightarrow\; P = 687.0\,\mathrm{d} = 1.881\,\mathrm{yr}.$$

From the third law, $a = P^{2/3} = 1.881^{2/3} = 1.524\,\mathrm{AU}$.

Problem 6.36 (Planetary configurations)**.** We are making an observation on 1 February 2022. Suppose that on this date Mars is at opposition and, at the same time, Jupiter is at western quadrature (Figure 6.5). Take the synodic periods to be 780 days for Mars and 398.9 days for Jupiter, Jupiter's orbital radius to be 5.20 AU, and the orbits to be circular.

(a) Determine the date of the next conjunction of Mars.

(b) Determine the date of the next opposition of Jupiter.

(c) When would Mars and Jupiter next be at opposition at the same time, so that the Sun, the Earth, Mars and Jupiter lie in a line?

Solution.

(a) From opposition to conjunction the Sun-centred angle between the Earth and Mars changes by 180°, half a synodic period: 390 days. 1 February 2022 plus 390 days is 26 February 2023.

(b) At quadrature the angle at the Earth is 90°, so the Sun-centred angle between the Earth and Jupiter is $\cos^{-1}(1/5.20) = 78.9°$; at opposition it is zero. The time needed is $(78.9/360) \times 398.9 = 87$ days, which gives 29 April 2022. (It is *not* a quarter of the synodic period: the elongation changes by 90°, but the Sun-centred angle by only 78.9°. After western quadrature a superior planet is heading towards opposition; after eastern quadrature, towards conjunction.)

(c) Mars is at opposition at times $t = n \times 780$ days after 1 February 2022, and Jupiter at $t = 87 + m \times 398.9$ days, with n and m whole numbers. We need the two to coincide. Listing the oppositions of Mars and finding how far each falls from the nearest opposition of Jupiter:

n	t (years)	nearest Jupiter opposition
1	2.1	105 days away
7	14.9	187 days away
16	34.2	26 days away
17	36.3	9 days away
18	38.4	9 days away

The two never coincide exactly, but after 17 or 18 synodic periods of Mars, about 36 to 38 years (around 2058 and 2060 on these assumptions), the oppositions fall within about nine days of each other and the four bodies are very nearly in line. Because 43 synodic periods of Jupiter almost equal 22 of Mars, the whole pattern of near-alignments repeats about every 47 years.

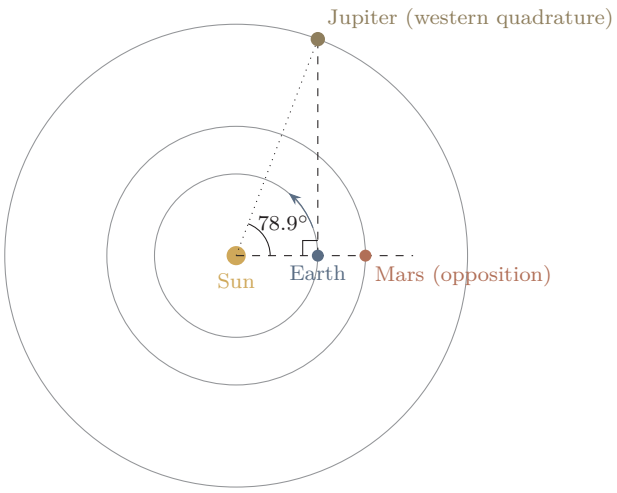


Figure 6.5: The configuration of the problem, seen from the north (not to scale; the angle at the Sun is drawn smaller than its true value because Jupiter's orbit is shown too small). The planets move anticlockwise, and the Earth, moving fastest, is catching up with Jupiter.

Problem 6.37 (Asteroid in an elliptical orbit around the Sun)**.** A small asteroid orbits the Sun with semi-major axis 2.5 AU and eccentricity 0.30 (Figure 6.6).

(a) Determine the orbital period.

(b) Find its orbital speed at perihelion.

(c) Find its orbital speed at aphelion.

(d) Compute its total orbital energy if its mass is $5 \times 10^{10}\,\mathrm{kg}$.

Solution.

(a) $T = a^{3/2} = 2.5^{3/2} = 3.95$ years.

(b) $r_p = a(1-e) = 1.75\,\mathrm{AU} = 2.618 \times 10^{11}\,\mathrm{m}$ and $a = 3.740 \times 10^{11}\,\mathrm{m}$. With $GM_\odot = 1.327 \times 10^{20}\,\mathrm{m^3\,s^{-2}}$, the vis-viva equation gives

$$v_p = \sqrt{1.327 \times 10^{20}\left(\frac{2}{2.618 \times 10^{11}} - \frac{1}{3.740 \times 10^{11}}\right)} = 25.7\,\mathrm{km\,s^{-1}}.$$

(c) $r_a = a(1+e) = 3.25\,\mathrm{AU} = 4.862 \times 10^{11}\,\mathrm{m}$, and in the same way $v_a = 13.8\,\mathrm{km\,s^{-1}}$. Check with the second law: $r_p v_p = 1.75 \times 25.7 = 45.0$ and $r_a v_a = 3.25 \times 13.8 = 44.9$, equal as they should be.

(d) The energy per unit mass is $-GM_\odot/2a = -(1.327\times10^{20})/(7.480\times10^{11}) = -1.77 \times 10^{8}\,\mathrm{J\,kg^{-1}}$, so

$$E = (5 \times 10^{10})(-1.77 \times 10^{8}) = -8.9 \times 10^{18}\,\mathrm{J}.$$

The energy is negative because the asteroid is bound to the Sun. The same value follows from $\frac{1}{2}v_p^2 - GM_\odot/r_p$ at perihelion.

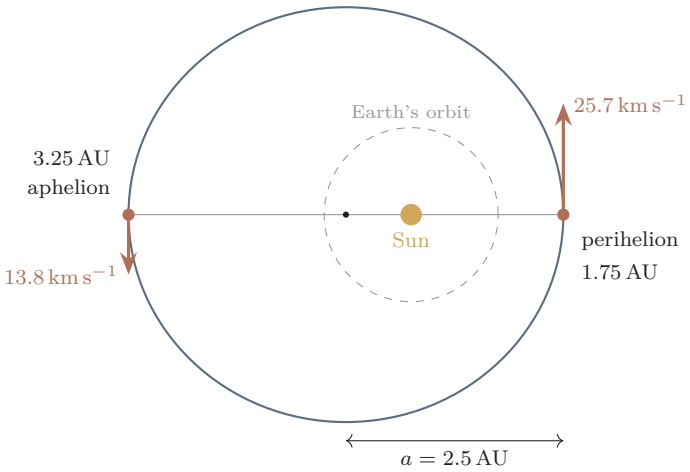


Figure 6.6: The orbit of the asteroid, to scale. The arrows show the velocity at perihelion and aphelion; their lengths are in proportion to the speeds.

Problem 6.38 (Mercury seen from Venus [adapted from USAAAO 2016, First Round, Q10 and Q11])**.** Mercury's orbit has semi-major axis 0.387 AU.

(a) Find Mercury's orbital speed when it is 0.340 AU from the Sun.

(b) Treating the orbits of Mercury and Venus (0.719 AU) as circles, find the greatest elongation of Mercury as seen from Venus.

Solution.

(a) $v = 29.8\sqrt{2/0.340 - 1/0.387} = 29.8 \times 1.816 = 54\,\mathrm{km\,s^{-1}}$.

(b) The line of sight from Venus is tangent to Mercury's orbit, so $\sin\epsilon_{\max} = 0.387/0.719$ and $\epsilon_{\max} = 32.6°$. Compare 22.8° as seen from the Earth: the closer the observer's own orbit, the larger the greatest elongation.

Problem 6.39 (The Earth's slightly eccentric orbit)**.** The Earth's orbit has eccentricity $e = 0.0167$. Find the ratio of the solar flux at perihelion to that at aphelion, and the Earth's orbital speed at each point. In which month is the Earth closest to the Sun, and why is that not the warmest time of year in Canada?

Solution. The distances are $r_p = 0.9833\,\mathrm{AU}$ and $r_a = 1.0167\,\mathrm{AU}$, so the flux ratio is $(r_a/r_p)^2 = (1.0167/0.9833)^2 = 1.069$: sunlight is about 7% stronger at perihelion. With $a = 1\,\mathrm{AU}$,

$$v_p = 29.78\sqrt{\frac{2}{0.9833} - 1} = 30.3\,\mathrm{km\,s^{-1}},$$
$$v_a = 29.78\sqrt{\frac{2}{1.0167} - 1} = 29.3\,\mathrm{km\,s^{-1}}.$$

Perihelion falls in early January. The seasons are caused by the tilt of the Earth's axis, not by its distance from the Sun: in January the northern hemisphere is tilted away from the Sun, and the low Sun and short days far outweigh the 7% gain in flux.

Problem 6.40 (Launch from the Moon [adapted from USAAAO 2016, First Round, Q4])**.** A probe is launched horizontally from the surface of the Moon (mass $7.44 \times 10^{22}\,\mathrm{kg}$, radius 1740 km) into an orbit whose farthest point is 7 lunar radii from the Moon's centre. What launch speed is needed? Compare it with the circular and escape speeds at the surface.

Solution. The launch point is the nearest point of the orbit, $r_p = R$, and $r_a = 7R$, so $a = 4R$. From the vis-viva equation,

$$v = \sqrt{GM\left(\frac{2}{R} - \frac{1}{4R}\right)} = \sqrt{\frac{1.75\,(4.966 \times 10^{12})}{1.74 \times 10^{6}}} = 2.23 \times 10^{3}\,\mathrm{m\,s^{-1}},$$

using $GM = (6.674 \times 10^{-11})(7.44 \times 10^{22}) = 4.966 \times 10^{12}\,\mathrm{m^3\,s^{-2}}$. The circular speed at the surface is $\sqrt{GM/R} = 1.69\,\mathrm{km\,s^{-1}}$ and the escape speed is $\sqrt{2}$ times

that, $2.39\,\mathrm{km\,s^{-1}}$. The answer lies between them, as it must for an elongated but bound orbit.

Problem 6.41 (On the way to geostationary orbit [adapted from USAAAO 2016, First Round, Q7])**.** A satellite is in a circular parking orbit 200 km above the Earth (mass 5.972×10^{24} kg, radius 6371 km). An engine burn puts it on a transfer ellipse whose apogee is at geostationary altitude, 35 786 km. Find the change of speed required and the time taken to reach apogee.

Solution. $GM = 3.986 \times 10^{14}\,\mathrm{m^3\,s^{-2}}$, $r_1 = 6571\,\mathrm{km}$, $r_2 = 42\,157\,\mathrm{km}$, and the transfer ellipse has $a_\mathrm{t} = (r_1 + r_2)/2 = 24\,364\,\mathrm{km}$.

$$v_\mathrm{c} = \sqrt{\frac{GM}{r_1}} = 7.79\,\mathrm{km\,s^{-1}}, \qquad v_p = \sqrt{GM\left(\frac{2}{r_1} - \frac{1}{a_\mathrm{t}}\right)} = 10.24\,\mathrm{km\,s^{-1}}.$$

The burn must add $\Delta v = 10.24 - 7.79 = 2.46\,\mathrm{km\,s^{-1}}$. The time to apogee is half the period of the ellipse, $\pi\sqrt{a_\mathrm{t}^3/GM} = 1.89 \times 10^4\,\mathrm{s} = 5.3$ hours.

6.6 Working with data

Problem 6.42 (Measuring the Hubble constant)**.** The table gives the distances and recession speeds of six galaxies.

d (Mpc)	20	45	70	110	150	200
v ($\mathrm{km\,s^{-1}}$)	1500	3050	5000	7600	10 700	13 900

(a) Plot v against d and find the Hubble constant from the slope of the best straight line through the origin.

(b) Estimate the uncertainty of your value.

(c) Find the corresponding Hubble time.

Solution.

(a) The points lie close to a straight line through the origin (Figure 6.7). For a line $v = H_0 d$ forced through the origin, the least-squares slope is

$$H_0 = \frac{\sum d_i v_i}{\sum d_i^2} = \frac{5.738 \times 10^6}{8.193 \times 10^4} = 70.0\,\mathrm{km\,s^{-1}\,Mpc^{-1}}.$$

A line drawn by eye on graph paper gives the same value to within a few per cent, and that is the method expected in an exam.

(b) The individual ratios v/d are 75.0, 67.8, 71.4, 69.1, 71.3 and 69.5. They scatter by about ± 3 around the mean, so the mean of six values is uncertain by about $3/\sqrt{6} \approx 1$. A reasonable answer is $H_0 = 70 \pm 1\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

(c) $t_H = 1/H_0$. Since $1\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ corresponds to 978 billion years, $t_H = 978/70 = 14.0$ billion years.

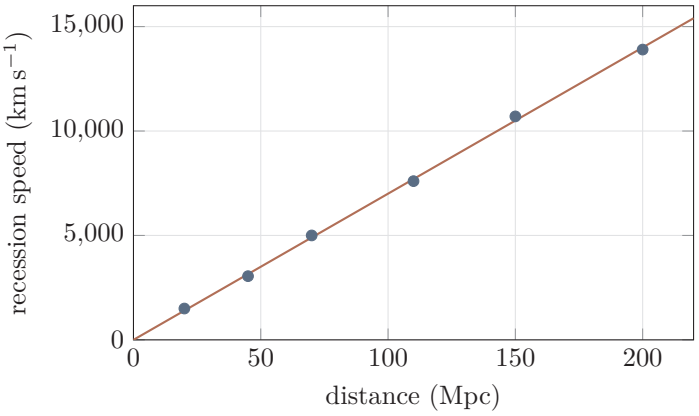


Figure 6.7: The data of the last problem with the best-fitting line through the origin, $v = 70.0\,d$.

Appendices

APPENDIX A

Mathematical Tools

This appendix collects the mathematics that the rest of the book takes for granted. None of it goes beyond a strong high-school course, but all of it must be fluent.

A.1 Powers of ten and logarithms

Astronomical quantities span an enormous range, so they are written in scientific notation, $a \times 10^n$ with $1 \le a < 10$. To multiply, multiply the numbers and add the exponents; to divide, divide and subtract.

The (base-10) logarithm of x is the power to which 10 must be raised to give x: if $x = 10^y$ then $\log_{10} x = y$. The rules used constantly with magnitudes are

$$\log_{10}(xy) = \log_{10} x + \log_{10} y, \qquad \log_{10}(x/y) = \log_{10} x - \log_{10} y,$$
$$\log_{10}(x^n) = n \log_{10} x, \qquad 10^{\log_{10} x} = x.$$

A few values are worth knowing by heart: $\log_{10} 2 = 0.301$, $\log_{10} 3 = 0.477$, $\log_{10} 5 = 0.699$. With them many magnitude problems can be checked without a calculator. For example, a factor of 2 in flux is $2.5 \times 0.301 = 0.75$ magnitudes, and a factor of 2 in distance is $5 \times 0.301 = 1.5$ magnitudes.

A.2 Angles

A full turn is 360° or 2π radians, so

$$1\,\mathrm{rad} = \frac{180^\circ}{\pi} = 57.296^\circ = 3438' = 206\,265''.$$

The radian is defined so that an arc of length s on a circle of radius r subtends an angle $\theta = s/r$ at the centre. This is why the small-angle formula $\theta = D/d$ of Chapter 1 works only in radians.

For small θ in radians,

$$\sin\theta \approx \theta, \qquad \tan\theta \approx \theta, \qquad \cos\theta \approx 1 - \tfrac{1}{2}\theta^2. \tag{A.1}$$

At $\theta = 10^\circ = 0.1745\,\mathrm{rad}$ the first of these is already accurate to half a per cent.

Angles are also measured in units of time: $24^\mathrm{h} = 360^\circ$, so

$$1^\mathrm{h} = 15^\circ, \qquad 1^\mathrm{m} = 15', \qquad 1^\mathrm{s} = 15'', \qquad 1^\circ = 4^\mathrm{m}.$$

Take care not to confuse minutes of time (m) with minutes of arc ($'$): they differ by a factor of 15.

A.3 Trigonometry

Basic identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan x = \frac{\sin x}{\cos x}$$
$$\sin(90^\circ - x) = \cos x \qquad \cos(90^\circ - x) = \sin x$$
$$\sin(180^\circ - x) = \sin x \qquad \cos(180^\circ - x) = -\cos x$$

The second line is used every time a formula of spherical trigonometry is rewritten in terms of latitude, declination or altitude. The third line is the reason an inverse sine is ambiguous: $\sin^{-1}$ returns an angle between -90° and 90°, but 180° minus that angle has the same sine.

Sum and double-angle formulas

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$$

Plane triangles

For a plane triangle with sides a, b, c opposite angles A, B, C:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \qquad c^2 = a^2 + b^2 - 2ab\cos C. \tag{A.2}$$

These are the limits of the spherical sine and cosine formulas when the triangle is very small.

A.4 Conic sections

The orbits of Chapter 5 are conic sections: the curves obtained by cutting a cone with a plane. All of them can be described by one equation in polar coordinates, with the origin at a focus:

$$r = \frac{p}{1 + e\cos\theta}, \tag{A.3}$$

where e is the eccentricity and p sets the size. The value of e decides the type: $e = 0$ is a circle, $0 < e < 1$ an ellipse, $e = 1$ a parabola and $e > 1$ a hyperbola.

The ellipse

An ellipse is the set of points for which the sum of the distances to two fixed points, the *foci*, is constant and equal to $2a$ (Figure A.1). Its longest diameter is the major axis, of length $2a$; the shortest is the minor axis, of length $2b$. Each focus is a distance $c = ae$ from the centre, and

$$b^2 = a^2 - c^2 = a^2(1 - e^2). \tag{A.4}$$

With the origin at the centre the equation of the ellipse is $x^2/a^2 + y^2/b^2 = 1$, and its area is πab. In Equation (A.3), $p = a(1 - e^2)$.

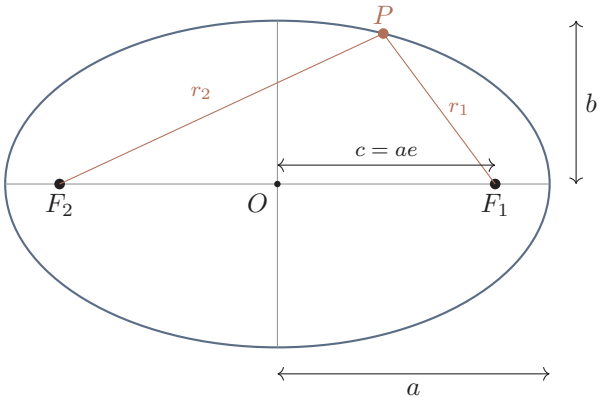


Figure A.1: An ellipse with semi-major axis a, semi-minor axis b and eccentricity $e = 0.8$. For every point P on the curve, $r_1 + r_2 = 2a$.

The parabola and the hyperbola

A parabola is the set of points equally distant from a focus and a fixed line. It is the limiting case of an ellipse whose second focus has moved to infinity; in Equation (A.3), $p = 2q$, where q is the distance from the focus to the nearest point of the curve.

A hyperbola is the set of points for which the *difference* of the distances to the two foci is constant and equal to $2a$. It has two branches, of which an orbiting body follows only the one that curves around the attracting focus, and far from the focus it approaches two straight lines called asymptotes. With the origin at the centre its equation is $x^2/a^2 - y^2/b^2 = 1$ with $b^2 = a^2(e^2 - 1)$, and in Equation (A.3) $p = a(e^2 - 1)$. The angle between the asymptotes gives the angle through which a passing body is deflected.

A.5 Polar coordinates and velocity

A point in a plane can be labelled by its distance r from the origin and the angle θ of its direction, measured anticlockwise from the x-axis:

$$x = r\cos\theta, \quad y = r\sin\theta \qquad \Longleftrightarrow \qquad r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}. \tag{A.5}$$

The velocity of a moving point has two components in these coordinates: a *radial* part, the rate at which r changes, and a *transverse* part $v_\perp = r\omega$, where ω is the rate at which θ changes (in radians per unit time):

$$v^2 = v_r^2 + v_\perp^2 = v_r^2 + (r\omega)^2. \tag{A.6}$$

The angular momentum of a body of mass m about the origin involves only the transverse part: $L = mrv_\perp = mr^2\omega$. At perihelion and aphelion the radial part vanishes, which is why $L = mrv$ there.

APPENDIX B

Uncertainties and Data Analysis

Every measurement has an uncertainty, and an answer without one is incomplete. The data-analysis questions of the olympiads reward a small number of habits, collected here.

B.1 Significant figures

Quote a final answer to the same number of significant figures as the least precise piece of data, usually two or three. Keep one or two extra figures in intermediate steps and round only at the end. An uncertainty is normally quoted to one significant figure (two if the first digit is 1), and the value is rounded to the same decimal place: $d = 20.0 \pm 0.8\,\mathrm{pc}$, not $20.0347 \pm 0.8123\,\mathrm{pc}$.

B.2 Combining uncertainties

Suppose x and y are measured independently with uncertainties Δx and Δy.

Calculation	Uncertainty of the result q
$q = x + y$ or $q = x - y$	$\Delta q = \sqrt{(\Delta x)^2 + (\Delta y)^2}$
$q = xy$ or $q = x/y$	$\dfrac{\Delta q}{q} = \sqrt{\left(\dfrac{\Delta x}{x}\right)^2 + \left(\dfrac{\Delta y}{y}\right)^2}$
$q = x^n$	$\dfrac{\Delta q}{q} = \lvert n \rvert \dfrac{\Delta x}{x}$
$q = \log_{10} x$	$\Delta q = 0.434\,\dfrac{\Delta x}{x}$

In words: for sums and differences, absolute uncertainties combine; for products and quotients, *relative* uncertainties combine; a power multiplies the relative uncertainty. Many exam mark schemes also accept the simpler rule of adding the uncertainties directly instead of in quadrature, which gives a slightly larger, safe value.

Two consequences are worth remembering.

- **Magnitudes.** Since $m = -2.5 \log_{10} b + \text{constant}$, an uncertainty in flux gives $\Delta m = 2.5 \times 0.434\,\Delta b/b = 1.09\,\Delta b/b$. A 5% error in flux is about 0.05 magnitudes.

- **Subtracting nearly equal numbers** destroys precision. If two quantities known to 1% differ by only 5%, their difference is uncertain by nearly 30%.

Example B.2.1 (Distance from a parallax)**.** A star has parallax $p = 0.050 \pm 0.002''$. Find its distance and uncertainty.

Solution. $d = 1/p = 20.0\,\mathrm{pc}$. Since $d = p^{-1}$, the relative uncertainty is the same as that of p: $0.002/0.050 = 4\%$. So $d = 20.0 \pm 0.8\,\mathrm{pc}$.

Example B.2.2 (Mass from an orbit)**.** The mass of a binary is found from $M = a^3/P^2$. If a is known to 2% and P to 1%, how well is M known?

Solution. The cube contributes $3 \times 2\% = 6\%$ and the square $2 \times 1\% = 2\%$. Combining, $\Delta M/M = \sqrt{6^2 + 2^2} = 6.3\%$. The uncertainty of the mass is dominated by the size of the orbit, because it enters as a cube.

B.3 Repeated measurements

If a quantity is measured N times, the best estimate is the mean,

$$\bar{x} = \frac{1}{N}\sum_i x_i, \tag{B.1}$$

and the scatter of the individual values is described by the standard deviation

$$s = \sqrt{\frac{1}{N-1}\sum_i (x_i - \bar{x})^2}. \tag{B.2}$$

The mean is known better than any single value: its uncertainty is $s/\sqrt{N}$. Four times as many measurements halve the uncertainty.

B.4 Straight-line graphs

Most data-analysis problems come down to drawing a straight line. If the relation between the measured quantities is not linear, change variables until it is:

- A power law $y = kx^n$ becomes a straight line when $\log y$ is plotted against $\log x$; the slope is the exponent n and the intercept is $\log k$. Kepler's third law is a line of slope $3/2$ in a plot of $\log T$ against $\log a$.
- An exponential $y = Ae^{-x/\tau}$ becomes a straight line when $\ln y$ is plotted against x; the slope is $-1/\tau$.
- Magnitudes are already logarithms: a period–luminosity relation is a straight line in a plot of M against $\log P$.

When drawing the graph, choose scales that make the data fill most of the page, label both axes with the quantity and its unit, plot every point, and draw the single straight line that passes as close as possible to all of them, with about as many points above as below. Find the slope from two points *on the line* that are far apart, not from two data points. To estimate the uncertainty of the slope, draw the steepest and the shallowest lines that are still reasonable fits and take half the difference of their slopes. Figure 6.7 in Chapter 6 is an example.

APPENDIX C

Constants and Data

Physical constants		
Speed of light	c	$2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{N\,m^{2}\,kg^{-2}}$
Stefan–Boltzmann constant	σ	$5.670 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
Wien's constant		$2.898 \times 10^{-3}\,\mathrm{m\,K}$
Boltzmann constant	k	$1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Planck constant	h	$6.626 \times 10^{-34}\,\mathrm{J\,s}$
Mass of hydrogen atom	m_{H}	$1.674 \times 10^{-27}\,\mathrm{kg}$
Units of distance and time		
Astronomical unit	AU	$1.496 \times 10^{11}\,\mathrm{m}$
Light-year	ly	$9.461 \times 10^{15}\,\mathrm{m}$
Parsec	pc	$3.086 \times 10^{16}\,\mathrm{m} = 206\,265\,\mathrm{AU}$
Sidereal year		$365.256\,\mathrm{d} = 3.156 \times 10^{7}\,\mathrm{s}$
Sidereal day		$23^{\mathrm{h}}56^{\mathrm{m}}04^{\mathrm{s}} = 86\,164\,\mathrm{s}$
Sun		
Mass	$M_{\odot}$	$1.989 \times 10^{30}\,\mathrm{kg}$
Radius	$R_{\odot}$	$6.957 \times 10^{8}\,\mathrm{m}$
Luminosity	$L_{\odot}$	$3.828 \times 10^{26}\,\mathrm{W}$
Effective temperature	$T_{\odot}$	$5772\,\mathrm{K}$
Apparent magnitude (V)	$m_{\odot}$	-26.74
Absolute magnitude (V)	$M_{\odot}$	4.83
Absolute bolometric magnitude		4.74
Solar constant		$1361\,\mathrm{W\,m^{-2}}$
Earth and Moon		
Mass of the Earth	$M_{\oplus}$	$5.972 \times 10^{24}\,\mathrm{kg}$
Radius of the Earth	$R_{\oplus}$	$6371\,\mathrm{km}$
Obliquity of the ecliptic	ε	23.44°
Mass of the Moon		$7.342 \times 10^{22}\,\mathrm{kg}$
Radius of the Moon		$1737\,\mathrm{km}$
Mean distance of the Moon		$384\,400\,\mathrm{km}$
Sidereal and synodic months		$27.32\,\mathrm{d}$ and $29.53\,\mathrm{d}$
Galaxy and cosmology		
Sun to Galactic centre		$\approx 8\,\mathrm{kpc}$
Hubble constant	H_0	$\approx 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$
Microwave background temperature		$2.725\,\mathrm{K}$

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About this book

This guide is written for students preparing for the Canadian Astronomy and Astrophysics Olympiad (CAAO). It assumes a solid background in high-school physics and mathematics, and nothing more.

Every new idea is followed at once by a worked example, many of them short questions from past papers of the International Olympiad on Astronomy and Astrophysics. Each chapter ends with a table of its formulas.

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Part I Theory

1 Starlight: distance, brightness and colour
2 Telescopes and detectors
3 Stars, galaxies and the expanding universe
4 Spherical astronomy
5 The solar system and celestial mechanics

Part II Practice

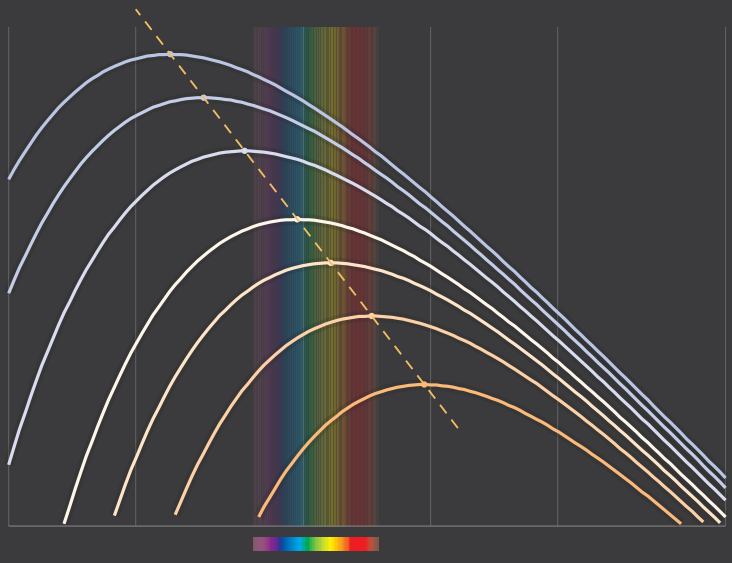
42 problems with full solutions

Appendices

Mathematical tools · Uncertainties and data analysis · Constants

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